Grassmannian Variables in Physics

by

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<u>Declaration</u>

I hereby declare that this thesis contains no material which has been accepted for the award of any other higher degree or graduate diploma in any tertiary institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by any other person, except where due reference is made in the text of this thesis.

Abstract

The five elements of this thesis are linked by the concept of Grassmannian variables. I begin with a brief introductory chapter discussing the general setting and then go on to deal with five topics each of which features anti-commuting co-ordinates in some guise.

Chapter 2 uses the conventional, space-time supersymmetry, admissible for relativistic field theories. In the same way that the Dirac equation can be regarded as a square root of the Klein-Gordon equation, I have obtained a square root of the Dirac equation. This equation involves spinor-valued superfields and is given in terms of two component Grassmannian spinors. It has a larger component field content than the Dirac equation, just as the Dirac equation has a larger content than the Klein-Gordon equation. After setting it up and solving the constraints for both the massless and massive cases, I have gone on for the massless case to solve the equation itself.

The next chapter is concerned with Grassmannian variables in the context of path integrals. Specifically, I have studied the derivation of the index of the twisted Dirac operator via a supersymmetric quantum mechanics and taken great care to establish how certain ambiguities in the path integral can arise and how they can be circumvented. As an aside I have also obtained the general expression for the index of fields of arbitrary spin from the Atiyah-Singer index theorem itself.

Chapter 4 uses anti-commuting co-ordinates as an appendage to the four commuting space-time co-ordinates. The Kaluza-Klein idea of force unification via general relativity is applied to a (4+N)-dimensional superspace. It is possible to give a consistent ansatz for a higher-dimensional metric which reproduces the standard model of elementary particles. I have considered the extension to grand unified theory and examined the SU(5) and SO(10) models, showing how the former is more natural and more economical than the latter within such a framework.

The consistent quantization of a gauge theory requires the inclusion of "ghost" fields having "wrong" spin and statistics. The resulting gauge-fixed, quantized theory is endowed with a BRST symmetry, which replaces the classical gauge invariance. This symmetry can be best understood when considered with a partner, the anti-BRST symmetry. The two are both supersymmetries as they mix commuting and anti-commuting fields and can therefore be formulated on a superspace with two Grassmannian co-ordinates. In chapter 5, I suggest that it is useful to do this in an Sp(2)-symmetric manner - that is with the ghosts and anti-ghosts and the BRST and anti-BRST symmetries themselves treated symmetrically - but that extending the symmetry group to OSp(4/2) is more of a hindrance than a help.

The final chapter of this thesis is concerned with a theory of massive non-abelian vector fields based on the Stueckelberg approach. Here renormalizability and unitarity are found to be conflicting requirements. Either one may be satisfied but not both. In particular, the violation of unitarity comes about either because of the failure of the BRST operator to be nilpotent or, diagrammatically, from the incomplete cancellation of the negative-norm ghost contributions.

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<u>Table of Contents</u>

Chapter 1
Introduction

The purpose of this chapter is just to give an outline of those areas in theoretical particle physics where Grassmannian or anticommuting variables occur, and then to mention the aspects which are relevant to each of the following chapters, where more detailed reviews will be given as appropriate.

Canonical quantization of spinorial fields in relativistic quantum field theory leads to anticommutation relations for these fields and Fermi-Dirac statistics for the associated particles. This connection between spin and statistics within relativistic quantum field theory is perhaps its greatest triumph, explaining as it does why electrons obey the Pauli exclusion principle while photons do not. By contrast one-loop effects are phenomenologically not as fundamental, though they do confirm the truth of quantum electrodynamics.

The $h \to \infty$ limit of the anticommutation relations leads to the consideration of totally anticommuting or Grassmannian fields. In the path integral approach to quantum field theory there is then the necessity of a definition of integration over such Grassmannian variables. This was supplied by Berezin - it is, for a Grassmannian variable θ,

$$\int d\theta \, (\theta a + b) = a \, .$$

In this thesis the convention for the ordering of multiple integrals will be such that

$$\int d\theta^1 ... d\theta^N \, \theta^N ... \theta^1 = 1 \, .$$

The result for gaussian integrals which follows from Berezin's integration rule is

$$\int d^N\theta \, e^{-\theta^t A \theta} \propto \det^{\frac{1}{2}} A \, ,$$

rather than $\det^{-\frac{1}{2}} A$ as for integration over commuting variables, and this ensures agreement between the operator and the path integral formulations of quantum field theory.

Anticommuting fields in violation of the spin-statistics theorem can still occur within the formalism of a relativistic quantum field theory provided they do not appear in the asymptotic states of the theory, that is provided they are not physical. Indeed such fields are, in general, necessary in the covariant treatment of theories which classically have a local gauge symmetry; these fields are the ghost fields.

Of course, quantum mechanical models too can be considered with anticommuting fields. Then the strictures of the spin-statistics theorem no longer apply.

A symmetry between commuting and anticommuting fields is called a supersymmetry. A consequence of the spin-statistics theorem is that the generators of any symmetry between physical commuting and anticommuting fields in relativistic quantum field theory must themselves carry a spinor representation. At the same time, the Coleman-Mandula no-go theorem says that the largest Lie algebra of generators of a relativistic quantum field theory consists of Lorentz scalars, apart from P_μ and $M_{\mu\nu}$, the generators of translations and rotations. Thus the generators of a supersymmetry cannot reside within a Lie algebra. They may, however, be found within the odd part of a graded Lie algebra or superalgebra, that is they must satisfy anticommutation relations amongst themselves. In this case they are still restricted so that they may only carry a spin $1/2$ representation of the Lorentz group, that is they must be of the form Q_α or $\overline{Q}_{\dot\alpha}$.

The simplest supersymmetry algebra admissible is

$$
\begin{aligned}
\{Q_\alpha,\overline{Q}_{\dot\beta}\} &= 2\sigma^\mu_{\alpha\dot\beta}P_\mu \\
\{Q_\alpha,Q_\beta\} = 0 &= \{\overline{Q}_{\dot\alpha}\overline{Q}_{\dot\beta}\} \\
[P_\mu,Q_\alpha] = 0 &= [P_\mu,\overline{Q}_{\dot\alpha}] \\
[P_\mu,P_\nu] &= 0
\end{aligned}
\tag{1}
$$

This may be extended by considering $Q_\alpha{}^A$, $Q_{\alpha B}$, for A,B = 1,...,N, with

$$\{Q^A_\alpha,\overline{Q}_{\dot\beta B}\} = 2\sigma^\mu_{\alpha\dot\beta}P_\mu\delta^A_B\,,$$

or by admitting central charges so that $\{Q_\alpha,Q_\beta\}$ and $\{\overline{Q}_{\dot\alpha},\overline{Q}_{\dot\beta}\}$ are non-zero. We shall refer to this sort of supersymmetry as the conventional or space-time supersymmetry (space-time since it mixes with the space-time transfomations). It is of course the supersymmetry of superstrings and supergravity.

Quantum mechanical models can be viewed as (0+1)-dimensional quantum field theories. In this light,for such models the supersymmetry algebra (1) becomes

$$\{Q,\overline{Q}\} = 2H,$$

where H is the hamiltonian operator of the model, with the other (anti)commutation relations vanishing.

The remaining possibility for anticommuting fields which was mentioned above leads to another sort of supersymmetry. Between gauge fields and their attendant ghost fields there is the BRST supersymmetry. This symmetry is fundamental in that it ensures that the ghost fields do not appear in the outgoing asymptotic states, so that they do not threaten unitarity. Also, as the remnant within the covariantly quantized Yang-Mills theory of the classical, local gauge symmetry, it implies identities which are used to prove renormalizability.

One way of constructing representations of a supersymmetry algebra is through a superfield construction on an appropriate superspace. A (d+N)-dimensional superspace is coordinatized by N Grassmannian variables, θ^m, as well as d ordinary x^μ. A superfield is a function on a superspace. Its dependence upon the Grassmannian coordinates is understood in terms of a power series with ordinary fields as coefficients, i.e.

$$F(x,\theta) = F(x) + \theta^m\chi_m(x) + \dots\,,$$

with the anticommuting nature of these coordinates ensuring that the series terminates at $\theta^1\cdots\theta^N$. The component fields $F(x),\chi_m(x),\dots$ are each taken to be commuting or anticommuting in such a way that the superfield as a whole is one or the other.

The superspace derivatives $\partial_\mu = \dfrac{\partial}{\partial x^\mu}$ and $\dfrac{\partial}{\partial\theta^m}$ act on such superfields.

Using

$$\left\{ \frac{\partial}{\partial\theta^m}, \frac{\partial}{\partial\theta^m} \right\} = 0 \text{ , etc.,}$$

they can be used to construct a representation of the supersymmetry algebra in the same way as position and orbital angular momentum can be represented in terms of ∂_μ and x^μ. Such a representation is then carried by the superfields and gives a representation of the algebra on the component fields through

$$\varepsilon QF(x,\theta) = \delta_\varepsilon F(x,\theta) = \delta_\varepsilon F(x) + \theta^m \delta_\varepsilon \chi_m(x) + \ldots = \varepsilon QF(x) + \theta^m \varepsilon Q \chi_m(x) + \ldots$$

In particular space-time supersymmetry can be represented on a (4+4)-dimensional superspace, $(x^\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}})$ by

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \ , \quad \bar{Q}^{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\alpha}}\partial_\mu \ , \quad P_\mu = -i\partial_\mu \ .$$

These operators obey

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = -\,2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu \text{ , etc.,}$$

with the change of sign here relative to (1) being necessary to ensure that the component fields carry a representation with the commutation relations (1).

A final note on complex conjugation. As the order of writing Grassmannian quantities is important a convention for the effect of complex conjugation on ordering is required. Throughout this thesis we have taken complex conjugation as reversing the order of all Grasmannian quantities, in this way it is compatible with more general hermitian conjugation.

This concludes our very brief review of the areas involving Grassmann variables. They are obviously very diverse and it is more appropriate to review the relevant aspects in each chapter as they arise; this is what we have done. The involvement of Grassmannian variables and/or supersymmetry in each of the following chapters is as follows.

Chapter 2 uses the conventional space-time supersymmetry, and in particular the superspace differential operators to construct a square root of the Dirac equation involving spinor-valued superfields. It begins with a review of superfields in N=1 supersymmetry.

The next chapter deals with the Atiyah-Singer index theorem. There is a derivation of the index for the twisted Dirac operator which uses path integrals within a supersymmetric quantum mechanics. We have examined this derivation, first locating the origin of certain ambiguities within the path integrals, and then circumventing them so as to obtain the correct final result. We found that the ambiguities do not have their origin in the Grassmannian integration rule. In the first part of this chapter, where we have reviewed the index theorem itself, we have also used it to derive the gravitational index theorems for fields of arbitrary spin.

Chapter 4 involves the use of superspaces, not to describe a supersymmetry but rather to, in analogy to the Kaluza-Klein scheme, to attempt a unification of gravity and Yang-Mills theory within the framework of supergravity. For this purpose we have reviewed the construction of supergravity. Specifically we have considered the grand unified models in this context.

The last two chapters are concerned with ghost fields and the BRST supersymmetry. In Chapter 5 we have considered the formulation of the extended-BRST supersymmetry on a (4+2)-dimensional superspace with an explicit Sp(2) symmetry of the Grassmannian coordinates maintained throughout. The conflicting nature of the requirements of renormalizability and unitarity in amassive Yang-Mills theory without Higgs is the subject of Chapter 6. Here the failure of unitarity which we demonstrate is bound up with the failure of the generator of the BRST symmetry to be nilpotent or the inability of preventing the ghost fields from contributing to the outgoing asymptotic states.

Work involved in Chapters 3,4,5, and 6 has been published in the following papers:

P.D.Jarvis and S.Twisk, Class.Quantum Grav. 4 (1987).

R.Delbourgo, S.Twisk and R.B.Zhang, Mod.Phys.Lett. A3 (1988) 1073.

R.Delbourgo, S.Twisk and G.Thompson, Int.J.Mod.Phys. A 3 (1988) 435.

S.Twisk and R.B.Zhang, Mod.Phys.Lett. A3 (1988) 1169.

Chapter 2
Space-Time Supersymmetry and a Square Root of the Dirac Equation

1. Introduction

Dirac initially found his equation by looking for a relativistic wave equation which would imply the Klein-Gordon equation, but which would be linear in $\partial/\partial t$ like the Schrodinger equation, and hence, hopefully, yield a positive definite probability density. So, in a sense, the Dirac equation was found as a square root of the Klein-Gordon equation. In so doing the argument of the equation became, instead of a scalar, a four-component spinor - appropriate for the desciption of electrons and other spin-$^1/_2$ particles. If, in turn, we seek to find a square root of the Dirac equation, then we must find an operator whose square is $i\not\partial$, just as $(i\not\partial)^2 = -\Box$. The argument of this operator would presumably be from an enlarged space, perhaps yielding some physically interesting multiplet of fields.

The spinorial differential operator $D = (D_\alpha, \bar{D}^{\dot\alpha})$ of supersymmetry satisfies the relation

$$-\frac{1}{2}\{D,D\} = i\not\partial$$

suggesting [1] that a square root of the Dirac operator might be found as an operator linear in $D_\alpha, \bar{D}^{\dot\alpha}$. In this chapter we find that this is indeed the case.

$D_\alpha, \bar{D}^{\dot\alpha}$ act of course on superfields which may carry an overall representation of the Lorentz group. We find that the equation

$$A\Psi(x,\theta,\bar\theta) = 0\,, \tag{1}$$

where $\Psi(x,\theta,\bar\theta) = \begin{pmatrix} \Phi_\alpha(x,\theta,\bar\theta) \\ \bar{X}^{\dot\alpha}(x,\theta,\bar\theta) \end{pmatrix}$ is a spinor-valued superfield satisfying the constraint

$$\begin{aligned} M_+\Phi_\alpha &\equiv (\tfrac{1}{2}D^\beta D_\beta + \bar{D}_{\dot\beta}\bar{D}^{\dot\beta})\Phi_\alpha(x,\theta,\bar\theta) = 0 \\ M\bar{X}^{\dot\alpha} &\equiv (D^\beta D_\beta + \tfrac{1}{2}\bar{D}_{\dot\beta}\bar{D}^{\dot\beta})\bar{X}^{\dot\alpha}(x,\theta,\bar\theta) = 0 \end{aligned} \tag{2}$$

and

$$A\Psi(x,\theta,\bar\theta) = \frac{1}{\sqrt{2}}\begin{pmatrix} D^\alpha\Phi_\alpha - \bar{D}_{\dot\alpha}\bar{X}^{\dot\alpha} \\ \bar{D}^{\dot\beta}\Phi_\beta + D_\beta\bar{X}^{\dot\beta} \end{pmatrix}\,,$$

is a square root of the Dirac equation,

$$i\not\partial\Psi = 0\,.$$

In terms of component fields (1) and (2) yield as propagating fields a Dirac spinor $\psi(x)$ and a complex vector field $A_\mu(x)$ satisfying the usual equations of motion

$$\begin{aligned} i\not\partial\psi(x) &= 0 \\ \partial^\mu A_{\mu\nu}(x) &= 0\,, \end{aligned} \tag{3}$$

where $A_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$.

We also give two square roots of the massive Dirac equation

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}\begin{pmatrix} \Psi(x,\theta,\bar\theta) \\ B(x,\theta,\bar\theta) \end{pmatrix} = \sqrt{m}\begin{pmatrix} \Psi(x,\theta,\bar\theta) \\ B(x,\theta,\bar\theta) \end{pmatrix}, \tag{4}$$

which also involves the bosonic superfield $B(x,\theta,\bar\theta)$ in the codomain of A. Here the spinor-superfield $\Psi(x,\theta,\bar\theta)$ may be constructed in two different ways, either by (2) or by

$$\begin{aligned} M_+\Phi_\alpha &= M\Phi_\alpha \\ M_-\bar X^{\dot\alpha} &= M\bar X^{\dot\alpha} \end{aligned} \tag{5}$$

where $M = \dfrac{\sqrt{2}}{1-\sqrt{2}}\, m$.

First we discuss briefly the superfield formulation of N=1 supersymmetry (in four dimensions). The notation and conventions used (basically those of [2]) are explained in the Appendix, where some useful identities are also given.

2. Superfields in N=1 Supersymmetry

Superfields are functions on superspace, $F(x,\theta,\bar\theta)$. They may take values in a space carrying some representation of the Lorentz group as well as of some internal symmetry group. Their $\theta,\bar\theta$ dependence should be understood in terms of their power series expansion into component fields,

$$\begin{aligned} F(x,\theta,\bar\theta) = {} & f(x) + \theta^\alpha\varphi_\alpha(x) + \bar\theta_{\dot\alpha}\bar\chi^{\dot\alpha} + \theta\theta m(x) + \bar\theta\bar\theta n(x) + \theta\sigma^\mu\bar\theta v_\mu(x) \\ & + \theta\theta\bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha}(x) + \bar\theta\bar\theta\theta^\alpha\psi_\alpha(x) + \theta\theta\bar\theta\bar\theta d(x) . \end{aligned}$$

Thus a superfield carrying an overall representation of the Lorentz group of spin s will, in general, contain component fields carrying spin up to s+1. This limits the number of superfields which are physically interesting; and indeed for flat superspace the only fields usually considered are those carrying only the trivial representation of the Lorentz group - scalar-valued superfields. However, constraints and the equations of motion may in other cases imply that the higher spin fields do not propagate anyway. This is the case for the spinor-valued superfield that we wil be considering.

As mentioned in chapter 1, superfields carry a representation of the supersymmetry algebra through the realization

$$\begin{aligned} Q_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \\ \bar Q_{\dot\alpha} &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \end{aligned} \tag{6}$$

$$\delta_\xi F(x,\theta,\bar\theta) = (\xi^\alpha Q_\alpha + \bar\xi_{\dot\alpha}\bar Q^{\dot\alpha})F(x,\theta,\bar\theta) .$$

In general, the representation that a superfield carries will not be irreducible. In order to descibe an irreducible representation of the supersymmetry algebra a superfield must be constrained in some way.

The operators

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu$$
$$\bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\,, \tag{7}$$

satisfying $\{D_\alpha,\bar D_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$, anticommute with Q_α, $\bar Q_{\dot\alpha}$,

$$\{D_\alpha,Q_\alpha\} = \{D_\alpha,\bar Q_{\dot\alpha}\} = \{\bar D_{\dot\alpha},Q_\alpha\} = \{\bar D_{\dot\alpha},\bar Q_{\dot\alpha}\} = 0\,.$$

Thus they can be used to place constraints on asuperfield in such a way that the constaint will be preserved under a supersymmetry transformation, i.e. in such a way that the constrained superfield will still carry a representation of the supersymmetry algebra.

On a complex scalar-valued superfield $S(x,\theta,\bar\theta)$, the first such constraint is

$$\bar D_{\dot\alpha}S(x,\theta,\bar\theta) = 0\,. \tag{8}$$

Writing $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$, we have

$$\bar D_{\dot\alpha}y^\mu = 0\,,\ \bar D_{\dot\alpha}\theta = 0$$

giving the general solution to (8)

$$\begin{aligned} S(x,\theta,\bar\theta) &= S(y,\theta)\\ &= A(y) + \sqrt{2}\,\theta\psi(y) + \theta\theta F(y)\\ &= A(x) + i\theta\sigma^\mu\bar\theta\partial_\mu A(x) - \tfrac{1}{4}\theta\theta\bar\theta\bar\theta\Box A(x) + \sqrt{2}\,\theta\psi(x) + \tfrac{i}{\sqrt{2}}\theta\theta\bar\theta\bar\sigma^\mu\partial_\mu\psi(x)\\ &\qquad + \theta\theta F(x)\,. \end{aligned}$$

This multiplet of fields $(A(x),\psi_\alpha(x),F(x))$ is known as the scalar or chiral multiplet and a superfield $S(x,\theta,\bar\theta)$ satisfting (8) as a scalar or chiral superfield, as is its conjugate $S(x,\theta,\bar\theta)$ which satisfies

$$D_\alpha\bar S(x,\theta,\bar\theta) = 0 \tag{9}$$

The appropriate free supersymmetric action is

$$\begin{aligned} &\int d^4x\int d^2\theta\int d^2\bar\theta\left(\bar SS + \tfrac{1}{2}mSS\delta(\bar\theta) + \tfrac{1}{2}m\bar S\bar S\delta(\theta)\right)\\ &\qquad = \int d^4x\left(-\bar A\Box A + i\partial_\mu\bar\psi\bar\sigma^\mu\psi + \bar FF + m(AF + \bar A\bar F - \tfrac{1}{2}\psi\psi - \tfrac{1}{2}\bar\psi\bar\psi)\right)\,. \end{aligned}$$

This is the Wess-Zumino model. From (6) the supersymmetry transformations are

$$\begin{aligned} \delta_\xi A &= \sqrt{2}\,\xi\psi\\ \delta_\xi\psi &= i\sqrt{2}\,\sigma^\mu\bar\xi\partial_\mu A + \sqrt{2}\,\xi F\\ \delta_\xi F &= i\sqrt{2}\,\bar\xi\bar\sigma^\mu\partial_\mu\psi\,. \end{aligned}$$

The field F(x) acts only as a multiplier field and may be eliminated through its equation of motion giving the action

$$\int d^4x\left(-\bar{A}\Box A - m^2\bar{A}A + i\partial_\mu\bar{\psi}\bar{\sigma}^\mu\psi - \frac{1}{2}m(\psi\psi + \bar{\psi}\bar{\psi})\right),$$

which is still invariant under a set of transformations mixing A and ψ. However, this set of transformations will only close upon the supersymmetry algebra if the equations of motion for A and ψ are used, that is the supersymmetry algebra only holds on shell.

The other important multiplet of fields in flat supersymmetry can also be formulated in terms of a constrained scalar-valued superfield. A scalar-valued superfield $V(x,\theta,\bar{\theta})$ satisfying the reality constraint,

$$V(x,\theta,\bar{\theta}) = \overline{V(x,\theta,\bar{\theta})}\ , \tag{10}$$

is known as a vector superfield. The appropriate free supersymmetric action is

$$\int d^4x\int d^2\theta\, d^2\bar{\theta}\left(\frac{1}{8}VD\bar{D}^2DV + m^2VV\right).$$

For m=0, this is invariant under the transformation

$$V \to V + \Phi + \Phi^\dagger$$

for a scalar field Φ. This invariance can be used to what is called the Wess-Zumino gauge where $V^3 = 0$. Then

$$V = -\theta\sigma^\mu\bar{\theta}v_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x)\ ,$$

with v_μ and D real, and the action becomes

$$\int d^4x\left(\frac{1}{2}D^2 - \frac{1}{4}v^{\mu\nu}v_{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\bar{\lambda}\right),$$

where $v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$, and the residual gauge transformation is just

$$v_\mu \to v_\mu + \partial_\mu a\ , \text{ with } a \text{ real.}$$

$(v_\mu(x), \lambda_\alpha(x), \bar{\lambda}^{\dot{\alpha}}(x), D(x))$ is known as the vector multiplet. By allowing V to be Lie algebra valued this model may also be generalized to describe supersymmetric Yang-Mills theories.

Other constraints involving higher orders of D, $\bar{D}$ may be imposed upon a scalar-valued superfield. While

$$D_\alpha S = 0\ ,\ \bar{D}_{\dot{\alpha}}S = 0$$

implies that S is independent of x,

$$D^2S = 0\ ,\ \bar{D}^2S = 0\ ,\ S = \bar{S}$$

may be imposed. This yields the linear multiplet $(C(x), \chi_\alpha(x), \bar{\chi}^{\dot{\alpha}}(x), A_\mu(x))$, with C and A_μ real and with $\partial^\mu A_\mu = 0$. Such higher order constraints are not much discussed (in [2],[3] or [4] for example), perhaps because they tend to yield multiplets equivalent to the chiral or the vector multiplets and because they tend to give constraints upon the component fields which involve derivatives - such constraints would have to be put into the action by hand.

Superfields carrying overall non-trivial represntations of the Lorentz group are also not much discussed, except as regards supergravity. Constraints such as (8)

$$\overline{D}_{\dot{\alpha}}S(x,\theta,\overline{\theta}) = 0$$

generalize immediately to other superfields as they do not mix with the overall representation. As stated in the introduction, we will consider spinor-valued superfields with constraints of the form

$$(aD_\alpha D^\alpha + b\overline{D}_{\dot{\alpha}}\overline{D}^{\dot{\alpha}})\Phi(x,\theta,\overline{\theta}) = \begin{cases} 0 \\ M\Phi(x,\theta,\overline{\theta}) \end{cases} \tag{12}$$

Spinor-valued superfields have been considered with chiral constraints. A spinor-valued superfield W_α, $\overline{W}^{\dot{\alpha}}$ satisfying

$$\overline{D}_{\dot{\alpha}}W_\alpha = 0\,,\ D_\alpha\overline{W}^{\dot{\alpha}} = 0 \ \text{and}\ D^\alpha W_\alpha = \overline{D}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}} \tag{13}$$

yields an alternative formulation of the supersymmetric gauge theory through, in the abelian case, for example, the solution

$$W_\alpha = -\tfrac{1}{4}\overline{D}\overline{D}D_\alpha V$$
$$\overline{W}^{\dot{\alpha}} = -\tfrac{1}{4}DD\overline{D}^{\dot{\alpha}}V$$

for a real scalar-valued superfield V. A chiral spinor-valued superfield has also been considered in [5] and shown, with a certain choice of action, to be equivalent to the linear multiplet above.

Supergravity, which I will discuss in Chapter 4, is based upon curved superspace admitting local supersymmetry transformations. The vielbein and connection generalize to superspace and are related by once again by a constraint on the torsion (not zero though, as in conventional gravity). In an appropriate supergauge the vielbein can be brought to

$$E_M^{\ A}(x,0,0) = \begin{bmatrix} e_\mu^{\ a}(x) & \tfrac{1}{2}\psi_\mu^{\ \alpha}(x) & \tfrac{1}{2}\overline{\psi}_{\mu\dot{\alpha}}(x) \\ 0 & \delta_m^{\ \alpha} & 0 \\ 0 & 0 & \delta_{\dot{\alpha}}^{\ \dot{m}} \end{bmatrix}$$

where $M = (\mu,m,\dot{m})$ are curved space and $A = (a,\alpha,\dot{\alpha})$ are flat space indices. The component fields shown $e_\mu{}^a(x)$, $\psi_\mu{}^\alpha(x)$ and $\overline{\psi}_{\mu\dot{\alpha}}(x)$ represent the graviton and its spin-$^3/_2$ super-partner the gravitino.

For supersymmetric theories based upon extended (N>1) supersymmetry or in higher dimensions, it is sometimes the case that there is only a component field formulation. In these cases the full spectrum of auxiliary fields necessary to realize the algebra off shell may not be known and, as a result, although the propagating fields and their equations of motion are known, the superfields in which they might be found are not properly understood.

3. A Square Root of the Dirac Equation

In the chiral representation the Dirac equation is

$$(i\partial\!\!\!/)\psi(x) \equiv \begin{pmatrix} 0 & i\sigma^{\mu}_{\beta\dot\alpha}\partial_\mu \\ i\bar\sigma^{\mu\dot\beta\alpha}\partial_\mu & 0 \end{pmatrix}\begin{pmatrix} \varphi_\alpha(x) \\ \bar\chi^{\dot\alpha}(x) \end{pmatrix} = m\begin{pmatrix} \varphi_\beta(x) \\ \bar\chi^{\dot\beta}(x) \end{pmatrix} \equiv m\psi(x) \qquad (14)$$

with square

$$-\Box\psi(x) = m^2\psi(x) .$$

The Dirac operator is off-diagonal

$$(i\partial\!\!\!/) = \begin{pmatrix} 0 & (i\partial\!\!\!/)_- \\ (i\partial\!\!\!/)_+ & 0 \end{pmatrix}$$

and is hermitian with respect to the inner product of Dirac spinors

$$\int dx\, \overline{\psi}_1(x)\psi_2(x) = \int dx \left(\overline{\varphi}_{1\dot\alpha}(x)\overline{\chi}_2^{\dot\alpha}(x) + \chi_1^{\alpha}(x)\varphi_{2\alpha}(x)\right)$$

and with respect to

$$\int dx\, \overline{\psi_1^c}(x)\psi_2(x)$$

which decomposes into the inner products

$$\int dx\, \varphi_1^{\alpha}(x)\varphi_{2\alpha}(x) \quad \text{and} \quad \int dx\, \overline{\chi}_{1\dot\alpha}(x)\overline{\chi}_2^{\dot\alpha}(x)$$

on the spaces of positive and negative chirality spinors respectively - that is

$$(i\partial\!\!\!/)_+^* = (i\partial\!\!\!/)_- .$$

In the next chapter these facts (or their Euclidean versions) will be exploited to draw an analogy between the Dirac operator acting between the spaces of positive and negative chirality spinors and a supersymmetry operator acting between bosonic and fermionic spaces in order to find the index of the Dirac operator. Here we seek an operator, A, which acts on a spinor-valued superfield, $\Psi(x,\theta,\overline{\theta})$, of space-time supersymmetry and for which

$$A^*A = i\partial\!\!\!/ .$$

Since

$$-\tfrac{1}{2}\{D^{\alpha},\overline{D}^{\dot\alpha}\} = (i\partial)_+$$

$$-\tfrac{1}{2}\{D_{\alpha},\overline{D}_{\dot\alpha}\} = (i\partial)_- ,$$

such an A might be linear in D_α, $\overline{D}_{\dot\alpha}$. Then $A\Psi(x,\theta,\overline{\theta})$ would be a bosonic superfield - that is a commuting tensorial superfield $B(x,\theta,\overline{\theta})$. The square root of the Dirac equation would be

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}\begin{pmatrix} \Psi(x,\theta,\overline{\theta}) \\ B(x,\theta,\overline{\theta}) \end{pmatrix} = \sqrt{m}\begin{pmatrix} \Psi(x,\theta,\overline{\theta}) \\ B(x,\theta,\overline{\theta}) \end{pmatrix} \qquad (15)$$

implying that

$$i\partial\!\!\!/\Psi(x,\theta,\overline{\theta}) = m\Psi(x,\theta,\overline{\theta}) .$$

Let

$$\langle \Psi_1, \Psi_2 \rangle = \int d^4x\, d^2\theta\, d^2\bar\theta\, \overline{\Psi}_1(x,\theta,\bar\theta)\Psi_2(x,\theta,\bar\theta) \tag{16}$$

be the inner product on the space of Dirac spinor-valued superfields,

$$\langle S_1, S_2 \rangle = \int d^4x\, d^2\theta\, d^2\bar\theta\, \overline{S}_1(x,\theta,\bar\theta)S_2(x,\theta,\bar\theta) \tag{17}$$

on scalar-valued superfields and

$$\langle V_1, V_2 \rangle = \int d^4x\, d^2\theta\, d^2\bar\theta\, \overline{V}^{\alpha}_{1\dot\alpha}(x,\theta,\bar\theta)V^{\dot\alpha}_{2\alpha}(x,\theta,\bar\theta) \tag{18}$$

on vector-valued superfields, $V^{\dot\alpha}_{\alpha} = (\sigma^\mu\varepsilon)^{\dot\alpha}_{\alpha}V_\mu$.

The operator

$$A\Psi \equiv A\begin{pmatrix}\Phi_\alpha \\ \overline{X}^{\dot\alpha}\end{pmatrix} = \frac{1}{\sqrt2}\begin{pmatrix} D^\alpha\Phi_\alpha - \overline{D}_{\dot\alpha}\overline{X}^{\dot\alpha} \\ \overline{D}^{\dot\beta}\Phi_\beta + D_\beta\overline{X}^{\dot\beta}\end{pmatrix} \tag{19}$$

has dual given by

$$\langle A\Psi, B\rangle = \langle \Psi, A^*B\rangle \tag{20}$$

where $B = \begin{pmatrix} S \\ V^{\dot\beta}_{\beta}\end{pmatrix}$.

Now on a superfield $F(x,\theta,\bar\theta)$

$$\overline{D_\alpha F} = \pm \overline{D}_{\dot\alpha}\overline{F}$$

according as F is commuting or anti-commuting, and similarly

$$\overline{\overline{D}_{\dot\alpha}F} = \pm D_\alpha\overline{F} ,$$

since the conjugation reverses the order of any factors as explained in the appendix. Also, since

$$D_\alpha(F_1F_2) = (D_\alpha F_1)F_2 \pm F_1(D_\alpha F_2) ,$$

the rule for integration by parts is

$$\int d^2\theta\, (D_\alpha F_1)F_2 = \mp \int d^2\theta\, F_1(D_\alpha F_2) ,$$

once again according as F_1 is bosonic or fermionic. Similarly, for $\overline{D}_{\dot\alpha}$

$$\int d^2\theta\, (\overline{D}_{\dot\alpha}F_1)F_2 = \mp \int d^2\theta\, F_1(\overline{D}_{\dot\alpha}F_2) .$$

Thus

$$\begin{aligned}\langle A\Psi, B\rangle &= \frac{1}{\sqrt2}\int d^4x\, d^2\theta\, d^2\bar\theta \left(\overline{(D_\alpha\Phi^\alpha - \overline{D}_{\dot\alpha}\overline{X}^{\dot\alpha})}S + \overline{(\overline{D}^{\dot\beta}\Phi_\beta + D_\beta\overline{X}^{\dot\beta})}V^{\dot\beta}_{\beta}\right) \\ &= \frac{1}{\sqrt2}\int d^4x\, d^2\theta\, d^2\bar\theta \left(-\overline{\Phi}_{\dot\alpha}\overline{D}^{\dot\alpha}S + X^\alpha D_\alpha S - \overline{\Phi}_{\dot\beta}D^\beta V^{\dot\beta}_{\beta} - X^\beta\overline{D}_{\dot\beta}V^{\dot\beta}_{\beta}\right)\end{aligned}$$

and so by (20)

$$A^*B = \frac{1}{\sqrt2}\begin{pmatrix} D_\alpha S - \overline{D}_{\dot\beta}V^{\dot\beta}_{\alpha} \\ -\overline{D}^{\dot\alpha}S - D^\beta V^{\dot\alpha}_{\beta}\end{pmatrix}. \tag{21}$$

For this choice of the operator A then

$$A^*A\Psi = \frac{1}{2}\begin{pmatrix} D_\alpha D^\beta\Phi_\beta - D_\alpha\overline{D}_{\dot\beta}\overline{X}^{\dot\beta} - \overline{D}_{\dot\beta}\overline{D}^{\dot\beta}\Phi_\alpha - \overline{D}_{\dot\beta}D_\alpha\overline{X}^{\dot\beta} \\ -\overline{D}^{\dot\alpha}D^\beta\Phi_\beta + \overline{D}^{\dot\alpha}\overline{D}_{\dot\beta}\overline{X}^{\dot\beta} - D^\beta\overline{D}^{\dot\alpha}\Phi_\beta - D^\beta D_\beta\overline{X}^{\dot\alpha} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2}(\frac{1}{2}D^\beta D_\beta + \overline{D}_{\dot\beta}\overline{D}^{\dot\beta})\Phi_\alpha - \frac{1}{2}\{D_\alpha,\overline{D}_{\dot\beta}\}\overline{X}^{\dot\beta} \\ -\frac{1}{2}(D^\beta D_\beta + \frac{1}{2}\overline{D}_{\dot\beta}\overline{D}^{\dot\beta})\overline{X}^{\dot\alpha} - \frac{1}{2}\{\overline{D}^{\dot\alpha},D^\beta\}\Phi_\beta \end{pmatrix},$$

where we have used the identities

$$\xi_\alpha\xi^\beta = -\frac{1}{2}\xi^\gamma\xi_\gamma\delta^\beta_\alpha, \quad \overline{\xi}^{\dot\alpha}\overline{\xi}_{\dot\beta} = -\frac{1}{2}\overline{\xi}_{\dot\gamma}\overline{\xi}^{\dot\gamma}\delta^{\dot\alpha}_{\dot\beta}.$$

Writing $M(a,b) = aD^\beta D_\beta + b\overline{D}_{\dot\beta}\overline{D}^{\dot\beta}$, we have

$$A^*A\Psi = \begin{pmatrix} -M(\frac{1}{4},\frac{1}{2})\Phi_\alpha + i\sigma^\mu_{\alpha\dot\beta}\partial_\mu\overline{X}^{\dot\beta} \\ i\overline{\sigma}^{\mu\dot\alpha\beta}\partial_\mu\Phi_\beta - M(\frac{1}{2},\frac{1}{4})\overline{X}^{\dot\alpha} \end{pmatrix}$$

$$= i\partial\!\!\!/\Psi - \begin{pmatrix} M(\frac{1}{4},\frac{1}{2}) & 0 \\ 0 & M(\frac{1}{2},\frac{1}{4}) \end{pmatrix}\begin{pmatrix} \Phi \\ \overline{X} \end{pmatrix} \tag{22}$$

That is A is the square root of the Dirac operator on the space of superfields satisfying

$$M(\tfrac{1}{4},\tfrac{1}{2})\Phi_\alpha = 0$$
$$M(\tfrac{1}{2},\tfrac{1}{4})\overline{X}^{\dot\alpha} = 0\,. \tag{23}$$

Note the similarity of these conditions to those for the linear multiplet of a scalar superfield (11). For a scalar superfield satisfying

$$M(a,b)S = 0\,,$$

with $a \neq b$, by imposing as well $S = S$, we have $M(b,a)S = 0$ and hence

$$DDS = 0,\ \overline{DD}S = 0,\ S = \overline{S}$$

which are the conditions (11). However, for a Weyl spinor Φ_α we may not impose the reality condition in four dimensions, and so cannot reach a simple generalization of the linear multiplet. Of course we may consider Ψ to be Majorana

$$\Psi = \begin{pmatrix} \Phi_\alpha \\ \overline{\Phi}^{\dot\alpha} \end{pmatrix},$$

but this does not imply any further conditions.

An alternative set of conditions to achieve a square root of the Dirac equation is

$$M(\tfrac{1}{4},\tfrac{1}{2})\Phi_\alpha = M\Phi_\alpha$$
$$M(\tfrac{1}{2},\tfrac{1}{4})\overline{X}^{\dot\alpha} = M\overline{X}^{\dot\alpha}\ . \qquad (24)$$

Then

$$A^*A\Psi = i\partial\!\!\!/\Psi - M\Psi$$

and the equation (15) implies

$$i\partial\!\!\!/\Psi = (m+M)\Psi\ . \qquad (25)$$

We now look at the solutions of the constraints (23) and (24).

4. Solution of the Constraints

As the operator

$$M(a,b) = aD^\alpha D_\alpha + b\overline{D}_{\dot\alpha}\overline{D}^{\dot\alpha}$$

does not interact with the overall index of a superfield, we may as well suppress this index - provided we keep the component fields on the right so that their commutativity is not important.

$$\begin{aligned}D^\alpha D_\alpha &= \varepsilon^{\alpha\beta}\left(\frac{\partial}{\partial\theta^\beta} + i\sigma^\mu_{\beta\dot\beta}\overline{\theta}^{\dot\beta}\partial_\mu\right)\left(\frac{\partial}{\partial\theta^\alpha} + i\sigma^\nu_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\partial_\nu\right)\\
&= \varepsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\beta}\frac{\partial}{\partial\theta^\alpha} + i\varepsilon^{\alpha\beta}\left(-\sigma^\nu_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\partial_\nu\frac{\partial}{\partial\theta^\beta} + \sigma^\mu_{\beta\dot\beta}\partial_\mu\frac{\partial}{\partial\theta^\alpha}\right) - \varepsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta}\sigma^\nu_{\alpha\dot\alpha}\tfrac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\overline{\theta\theta}\partial_\mu\partial_\nu\\
&= \varepsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\beta}\frac{\partial}{\partial\theta^\alpha} + 2i(\varepsilon\sigma^\mu)^\alpha_{\dot\alpha}\overline{\theta}^{\dot\alpha}\frac{\partial}{\partial\theta^\alpha}\partial_\mu + \overline{\theta\theta}\Box \qquad (26)\end{aligned}$$

and

$$\begin{aligned}\overline{D}_{\dot\alpha}\overline{D}^{\dot\alpha} &= \varepsilon^{\dot\alpha\dot\beta}\left(-\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\right)\left(-\frac{\partial}{\partial\overline{\theta}^{\dot\beta}} - i\theta^\beta\sigma^\nu_{\beta\dot\beta}\partial_\nu\right)\\
&= \varepsilon^{\dot\alpha\dot\beta}\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}\frac{\partial}{\partial\overline{\theta}^{\dot\beta}} + 2i\theta^\alpha(\sigma^\mu\varepsilon)_\alpha^{\dot\alpha}\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}\partial_\mu + \theta\theta\Box \qquad (27)\end{aligned}$$

Let the expansion in $\theta,\overline{\theta}$ of a superfield F be

$$\begin{aligned}F(x,\theta,\overline{\theta}) &= f(x) + \theta^\alpha\varphi_\alpha(x) + \overline{\theta}_{\dot\alpha}\overline{\chi}^{\dot\alpha}(x) + \theta\theta m(x) + \overline{\theta\theta}n(x) + \theta\sigma^\mu\overline{\theta}v_\mu(x) + \theta\theta\overline{\theta\lambda}(x)\\
&\quad + \overline{\theta\theta}\theta\psi(x) + \theta\theta\overline{\theta\theta}d(x) \qquad (28)\end{aligned}$$

Then

$$\begin{aligned}D^\alpha D_\alpha F(x,\theta,\overline{\theta}) &= -4m(x) - 4\overline{\theta\lambda}(x) - 4\overline{\theta\theta}d(x)\\
&\quad + 2i(\varepsilon\sigma^\mu)^\alpha_{\dot\alpha}\overline{\theta}^{\dot\alpha}(\partial_\mu\varphi_\alpha(x) + 2\theta_\alpha\partial_\mu m(x) + \sigma^\nu_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}\partial_\mu v_\nu(x) + 2\theta_\alpha\overline{\theta}_{\dot\beta}\partial_\mu\overline{\lambda}^{\dot\beta}(x))\\
&\quad + \overline{\theta\theta}\Box f(x) + \overline{\theta\theta}\theta\Box\varphi(x) + \theta\theta\overline{\theta\theta}\Box m(x)\end{aligned}$$

$$= -4m(x) - 2\bar\theta(2\bar\lambda(x) + i\bar\sigma^\mu\partial_\mu\varphi(x)) + \bar\theta\bar\theta(-4d(x) - 2i\partial^\mu v_\mu(x) + \Box f(x))$$
$$+ 4i\theta\sigma^\mu\bar\theta\partial_\mu m(x) + \bar\theta\bar\theta\theta(\Box\varphi(x) - 2i\sigma^\mu\partial_\mu\bar\lambda(x)) + \theta\theta\bar\theta\bar\theta\Box m(x) \quad (29)$$

and

$$\bar D_{\dot\alpha}\bar D^{\dot\alpha}F(x,\theta,\bar\theta) = -4n(x) - 4\theta\psi(x) - 4\theta\theta d(x)$$
$$+ 2i\theta^\alpha(\sigma^\mu\varepsilon)_\alpha^{\dot\alpha}(-\partial_\mu\bar\chi_{\dot\alpha}(x) - 2\bar\theta_{\dot\alpha}\partial_\mu n(x) - \theta^\beta\sigma^\nu_{\beta\dot\alpha}\partial_\mu v_\nu(x) - 2\bar\theta_{\dot\alpha}\theta^\beta\partial_\mu\psi_\beta(x))$$
$$+ \theta\theta\Box f(x) + \theta\theta\bar\theta\Box\,\bar\chi(x) + \theta\theta\bar\theta\bar\theta\Box n(x)$$
$$= -4n(x) - 2\theta(2\psi(x) + i\sigma^\mu\partial_\mu\bar\chi(x)) + \theta\theta(-4d(x) + 2i\partial^\mu v_\mu(x) + \Box f(x))$$
$$- 4i\theta\sigma^\mu\bar\theta\partial_\mu n(x) + \theta\theta\bar\theta(\Box\bar\chi(x) - 2i\bar\sigma^\mu\partial_\mu\psi(x)) + \theta\theta\bar\theta\bar\theta\Box n(x)\ . \quad (30)$$

Thus

$$M(a,b)F(x,\theta,\bar\theta) = MF(x,\theta,\bar\theta) \quad (31)$$

implies that

$$-4am(x) - 4bn(x) = Mf(x) \quad (31a)$$
$$-2b(2\psi(x) + i\sigma^\mu\partial_\mu\bar\chi(x)) = M\varphi(x) \quad (31b)$$
$$-2a(2\bar\lambda(x) + i\bar\sigma^\mu\partial_\mu\varphi(x)) = M\bar\chi(x) \quad (31c)$$
$$b(-4d(x) + 2i\partial^\mu v_\mu(x) + \Box f(x)) = Mm(x) \quad (31d)$$
$$a(-4d(x) - 2i\partial^\mu v_\mu(x) + \Box f(x)) = Mn(x) \quad (31e)$$
$$-4ia\partial_\mu m(x) - 4ib\partial_\mu n(x) = Mv_\mu(x) \quad (31f)$$
$$b(\Box\bar\chi(x) - 2i\bar\sigma^\mu\partial_\mu\psi(x)) = M\bar\lambda(x) \quad (31g)$$
$$a(\Box\varphi(x) - 2i\bar\sigma^\mu\partial_\mu\bar\lambda(x)) = M\psi(x) \quad (31h)$$
$$a\Box m(x) + b\Box n(x) = Md(x) \quad (31i)$$

We consider first the case that $M = 0$, $a,b \neq 0$. Then (31a) and (31f) imply (31i) and that

$$m(x) = c\text{ , a constant,}$$

and

$$n(x) = -\frac{a}{b}c\ .$$

(31b)

$$\psi(x) = -\frac{i}{2}\sigma^\mu\partial_\mu\bar\chi(x)$$

implies (31g). (31c)

$$\bar\lambda(x) = -\frac{i}{2}\bar\sigma^\mu\partial_\mu\varphi(x)$$

implies (31h). (31d) + (31e) imply that

$$d(x) = \frac{1}{4}\Box f(x)$$

and (31d) - (31e) that

$$\partial^\mu v_\mu(x) = 0 .$$

So the general solution to

$$M(a,b)F(x,\theta,\bar\theta) = 0$$

is

$$F(x,\theta,\bar\theta) = f(x) + \theta\varphi(x) + \bar\theta\bar\chi(x) + \theta\theta c - \frac{a}{b}\,\bar\theta\bar\theta c + \theta\sigma^\mu\bar\theta v_\mu(x) - \frac{i}{2}\theta\,\theta\bar\theta\bar\sigma^\mu\partial_\mu\varphi(x)$$
$$- \frac{i}{2}\,\bar\theta\bar\theta\theta\sigma^\mu\partial_\mu\bar\chi(x) + \frac{1}{4}\theta\,\theta\bar\theta\bar\theta\Box f(x) \,, \qquad (32)$$

where $\partial^\mu v_\mu(x) = 0$.

On the other hand, when $M \neq 0$, $a,b \neq 0$, (31a), (31d-f) and (31i) imply that

$$n(x) = -\frac{M}{4b} f(x) - \frac{a}{b} m(x),$$
$$v_\mu(x) = i\partial_\mu f(x) + \frac{8i}{M} a\partial_\mu m(x),$$
$$d(x) = \frac{M^2}{64ab} f(x) \qquad (33)$$

and

$$\Box f(x) = \frac{M^2}{16ab} f(x),$$
$$\Box m(x) = -\frac{M^2}{16ab} m(x).$$

The remaining equations imply that

$$\bar\lambda(x) = \frac{i}{2}\bar\sigma^\mu\partial_\mu\varphi(x) \,,$$
$$\bar\chi(x) = -\frac{4ia}{M}\bar\sigma^\mu\partial_\mu\varphi(x) \,, \qquad (34)$$
$$\psi(x) = -\frac{M}{8b}\varphi(x)$$

and

$$\Box\,\varphi(x) = -\frac{M^2}{16ab}\varphi(x) \,.$$

Thus the general solution to

$$(aD^\alpha D_\alpha + b\bar D_{\dot\alpha}\bar D^{\dot\alpha})F(x,\theta,\bar\theta) = MF(x,\theta,\bar\theta)$$

for $a,b,M \neq 0$ is

$$F(x,\theta,\bar\theta) = f(x) + \theta\varphi(x) - \frac{4ia}{M}\bar\theta\bar\sigma^\mu\partial_\mu\varphi(x) + \theta\theta m(x) + \bar\theta\bar\theta\left(-\frac{M}{4b} f(x) - \frac{a}{b} m(x)\right)$$
$$+ \theta\sigma^\mu\bar\theta\left(\frac{8i}{M} a\partial_\mu m(x) + i\partial_\mu f(x)\right) + \frac{i}{2}\theta\,\theta\bar\theta\bar\sigma^\mu\partial_\mu\varphi(x) - \frac{M}{8b}\bar\theta\bar\theta\theta\varphi(x) \qquad (35)$$
$$+ \frac{M^2}{64ab}\,\theta\theta\bar\theta\bar\theta f(x) \,,$$

with

$$\Box F(x,\theta,\bar\theta) = -\frac{M^2}{16ab} F(x,\theta,\bar\theta) \,.$$

The supersymmetry transformations of the component fields in both cases can be found through (6)

$$\delta_\xi F(x,\theta,\bar\theta) = (\xi^\alpha Q_\alpha + \bar\xi_{\dot\alpha}\bar Q^{\dot\alpha})F(x,\theta,\bar\theta)\ .$$

For the multiplet of (32), $(f(x),\varphi_\alpha(x),\bar\chi^{\dot\alpha}(x),c,v_\mu(x))$ with $\partial^\mu v_\mu(x) = 0$, they are

$$\begin{aligned}
\delta_\xi f(x) &= \xi\varphi(x) + \bar\xi\bar\chi(x)\\
\delta_\xi \varphi(x) &= 2\xi c + \sigma^\mu\bar\xi(v_\mu(x) + i\partial_\mu f(x))\\
\delta_\xi \bar\chi(x) &= -\frac{2a}{b}\bar\xi m - \bar\sigma^\mu\xi(v_\mu(x) + i\partial_\mu f(x))\\
\delta_\xi c &= 0\\
\delta_\xi v_\mu(x) &= \frac{i}{2}\xi(\sigma^\nu\bar\sigma^\mu - \sigma^\mu\bar\sigma^\nu)\partial_\nu\varphi(x) + \frac{i}{2}\bar\xi(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)\partial_\nu\bar\chi(x)
\end{aligned} \tag{36}$$

For the multiplet of (35), $(f(x),\varphi_\alpha(x),m(x))$ with $\Box = -\frac{M^2}{16ab}$, they are

$$\begin{aligned}
\delta_\xi f(x) &= \xi\varphi(x) - \frac{4ia}{M}\bar\xi\bar\sigma^\mu\partial_\mu\varphi(x)\\
\delta_\xi \varphi(x) &= 2\xi m(x) + \sigma^\mu\bar\xi\left(2i\partial_\mu f(x) + \frac{8ia}{M}\partial_\mu m(x)\right)\\
\delta_\xi m(x) &= i\bar\xi\bar\sigma^\mu\partial_\mu\varphi(x)
\end{aligned} \tag{37}$$

5. Component Field Content

Using the results of the previous section, we can now write down the spinor-valued superfields upon which the operator A is indeed the square root of the Dirac operator.

For the massless case

$$A\Psi(x,\theta,\bar\theta) \equiv \frac{1}{\sqrt2}\begin{pmatrix} D^\alpha\Phi_\alpha - \bar D_{\dot\alpha}\bar X^{\dot\alpha} \\ \bar D^{\dot\beta}\Phi_\beta + D_\beta\bar X^{\dot\beta}\end{pmatrix} = 0 \tag{38}$$

implies that

$$i\partial\!\!\!/\Psi(x,\theta,\bar\theta) = 0\ ,$$

if

$$\begin{aligned}
\Phi_\alpha(x,\theta,\bar\theta) = \varphi_\alpha(x) &+ \theta^\beta m_{\beta\alpha}(x) + (\bar\theta\bar\sigma^\mu\varepsilon)_\alpha v_\mu(x) + \theta\theta\psi_\alpha - \tfrac{1}{2}\bar\theta\bar\theta\psi_\alpha + \theta\sigma^\mu\bar\theta\lambda_{\mu\alpha}(x)\\
&- \tfrac{i}{2}\theta\,\theta\bar\theta_{\dot\beta}\bar\sigma^{\mu\dot\beta\beta}\partial_\mu m_{\beta\alpha}(x) - \tfrac{i}{2}\bar\theta\bar\theta(\theta\sigma^\mu\bar\sigma^\nu\varepsilon)_\alpha\partial_\mu v_\nu(x) + \tfrac{1}{4}\theta\,\theta\bar\theta\bar\theta\Box\varphi_\alpha(x)
\end{aligned} \tag{39}$$

and

$$\overline{X}^{\dot\alpha}(x,\theta,\overline{\theta}) = \overline{\chi}^{\dot\alpha}(x) + (\theta\sigma^\mu\varepsilon)^{\dot\alpha} w_\mu(x) + \overline{\theta}_{\dot\beta} n^{\dot\beta\dot\alpha}(x) + \theta\theta\overline{\varpi}^{\dot\alpha} - 2\overline{\theta\theta}\,\overline{\varpi}^{\dot\alpha} + \theta\sigma^\mu\overline{\theta}\,\overline{\xi}_\mu^{\dot\alpha}(x)$$
$$- \tfrac{i}{2}\,\theta\theta(\overline{\theta}\overline{\sigma}^\mu\sigma^\nu\varepsilon)^{\dot\alpha}\partial_\mu w_\nu(x) - \tfrac{i}{2}\,\overline{\theta\theta}\theta^\beta\sigma^\mu_{\beta\dot\beta}\partial_\mu n^{\dot\beta\dot\alpha}(x) + \tfrac{1}{4}\,\theta\,\theta\overline{\theta\theta}\Box\overline{\chi}^{\dot\alpha}(x)\,, \quad (40)$$

where $\partial^\mu\lambda_{\mu\alpha}(x) = 0$, $\partial^\mu\overline{\xi}_\mu^{\dot\alpha}(x) = 0$ and ψ_α, $\overline{\varpi}^{\dot\alpha}$ are constant spinors.

For the massive case

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}\begin{pmatrix} \Psi(x,\theta,\overline{\theta}) \\ B(x,\theta,\overline{\theta}) \end{pmatrix} = \sqrt{m}\begin{pmatrix} \Psi(x,\theta,\overline{\theta}) \\ B(x,\theta,\overline{\theta}) \end{pmatrix}$$

implies that

$$i\partial\!\!\!/\Psi(x,\theta,\overline{\theta}) = m\Psi(x,\theta,\overline{\theta})$$

if $\Psi(x,\theta,\overline{\theta})$ is given by (38) and (39), or that

$$i\partial\!\!\!/\Psi(x,\theta,\overline{\theta}) = (m+M)\Psi(x,\theta,\overline{\theta}) \quad (41)$$

if

$$\Phi_\alpha(x,\theta,\overline{\theta}) = \varphi_\alpha(x) - \tfrac{2i}{M}\theta^\beta\sigma^\mu_{\beta\dot\beta}\partial_\mu v^{\dot\beta}_\alpha(x) + \overline{\theta}_{\dot\beta}v^{\dot\beta}_\alpha(x) + \theta\theta\psi_\alpha(x) - \tfrac{1}{2}\overline{\theta\theta}(\psi_\alpha(x) + M\varphi_\alpha(x))$$
$$+ \theta\sigma^\mu\overline{\theta}(\tfrac{2i}{M}\partial_\mu\psi_\alpha(x) + i\partial_\mu\varphi_\alpha(x)) - \tfrac{M}{2}\theta\,\theta\overline{\theta}_{\dot\beta}v^{\dot\beta}_\alpha(x) + \tfrac{i}{2}\,\overline{\theta\theta}\theta^\beta\sigma^\mu_{\beta\dot\beta}\partial_\mu v^{\dot\beta}_\alpha(x)$$
$$+ \tfrac{M^2}{8}\theta\,\theta\overline{\theta\theta}\varphi_\alpha(x) \quad (42)$$

and

$$\overline{X}^{\dot\alpha}(x,\theta,\overline{\theta}) = \overline{\chi}^{\dot\alpha}(x) + \theta^\beta w^{\dot\alpha}_\beta(x) - \tfrac{2i}{M}\overline{\theta}_{\dot\beta}\overline{\sigma}^{\mu\dot\beta\beta}\partial_\mu w^{\dot\alpha}_\beta(x) + \theta\theta\overline{\varpi}^{\dot\alpha}(x) - \overline{\theta\theta}(M\overline{\chi}^{\dot\alpha}(x) + 2\overline{\varpi}^{\dot\alpha}(x))$$
$$+ \theta\sigma^\mu\overline{\theta}(\tfrac{4i}{M}\partial_\mu\overline{\varpi}^{\dot\alpha}(x) + i\partial_\mu\overline{\chi}^{\dot\alpha}(x)) + \tfrac{i}{2}\theta\,\theta\overline{\theta}_{\dot\beta}\overline{\sigma}^{\mu\dot\beta\beta}\partial_\mu w^{\dot\alpha}_\beta(x) - \tfrac{M}{2}\overline{\theta\theta}\theta^\beta w^{\dot\alpha}_\beta(x)$$
$$+ \tfrac{M^2}{8}\theta\,\theta\overline{\theta\theta}\,\overline{\chi}^{\dot\alpha}(x) \quad (43)$$

with

$$\Box\Psi(x,\theta,\overline{\theta}) = -\frac{M^2}{2}\Psi(x,\theta,\overline{\theta}) \quad (44)$$

In the latter case (40) and (43) imply that

$$(m+M)^2 = \frac{M^2}{2} \quad \text{or} \quad m = M(-1\pm\frac{1}{\sqrt{2}})\,,$$

and

$$i\partial\!\!\!/\Psi(x,\theta,\overline{\theta}) = \pm\frac{M}{\sqrt{2}}\Psi(x,\theta,\overline{\theta})\,.$$

The field content exhibited above may not all be propagating. In order to find the real physical content of our square root of the Dirac equation we must resolve the equations of motion for the component fields, in the process eliminating any auxiliary fields.

As yet we have said nothing about the bosonic superfield $B(x,\theta,\overline{\theta})$. It obeys the squared equation

$$AA^*B(x,\theta,\bar\theta) = mB(x,\theta,\bar\theta)\ , \tag{45}$$

which is

$$\begin{pmatrix} M(1,1)S + [\bar D_{\dot\beta}, D^\beta]V^{\dot\beta}_\beta \\ [\bar D_{\dot\beta}, D^\beta]S + \frac{1}{2}M(1,1)V^{\dot\beta}_\beta \end{pmatrix} = 2m\begin{pmatrix} S \\ V^{\dot\beta}_\beta \end{pmatrix}, \tag{46}$$

$$M(1,1) = D^\alpha D_\alpha + \bar D_{\dot\alpha}\bar D^{\dot\alpha}\ .$$

This equation still involves θ-derivatives, even if we impose $M(1,1)B = MB$. This is perhaps not unreasonable as a bosonic field should obey a second order equation involving m^2. In any case, we can ignore $B(x,\theta,\bar\theta)$ in consideration of the massless equation for $\Psi(x,\theta,\bar\theta)$, (38).

With Ψ given by (39), (40)

$$D^\alpha\Phi_\alpha(x,\theta,\bar\theta) - \bar D_{\dot\alpha}\bar X^{\dot\alpha}(x,\theta,\bar\theta)$$

$$= \varepsilon^{\alpha\beta}\Big(m_{\beta\alpha}(x) + 2\theta_\beta\psi_\alpha + \sigma^\mu_{\beta\dot\beta}\bar\theta^{\dot\beta}\lambda_{\mu\alpha}(x) - i\theta_\beta\bar\theta_{\dot\beta}\bar\sigma^{\mu\dot\beta\beta}\partial_\mu m_{\beta\alpha}(x) - \tfrac{i}{2}\overline{\theta\theta}(\sigma^\mu\bar\sigma^\nu\varepsilon)_{\beta\alpha}\partial_\mu v_\nu(x)$$

$$+ \tfrac{1}{2}\theta_\beta\overline{\theta\theta}\Box\varphi_\alpha(x)\Big) + i(\varepsilon\sigma^\rho)^\alpha_{\dot\alpha}\bar\theta^{\dot\alpha}\Big(\partial_\rho\varphi_\alpha(x) + \theta^\beta\partial_\rho m_{\beta\alpha}(x) + \bar\theta_{\dot\beta}(\bar\sigma^\mu\varepsilon)^{\dot\beta}_\alpha\partial_\rho v_\mu(x)$$

$$- \bar\theta\bar\sigma^\mu\theta\partial_\rho\lambda_{\mu\alpha}(x) - \tfrac{i}{2}\theta\theta\bar\theta_{\dot\beta}\bar\sigma^{\mu\dot\beta\beta}\partial_\mu\partial_\rho m_{\beta\alpha}(x)\Big)$$

$$+ \varepsilon_{\dot\beta\dot\alpha}n^{\beta\dot\alpha}(x) + 4\bar\theta_{\dot\alpha}\bar\omega^{\dot\alpha} - \theta\sigma^\mu\bar\xi_\mu(x) - \tfrac{i}{2}\theta\theta\varepsilon_{\dot\beta\dot\alpha}\bar\sigma^{\mu\dot\beta\beta}(\sigma^\nu\varepsilon)_\beta^{\dot\alpha}\partial_\mu w_\nu(x) + i\bar\theta_{\dot\alpha}\theta^\beta\sigma^\mu_{\beta\dot\beta}\partial_\mu n^{\dot\beta\dot\alpha}(x)$$

$$- \tfrac{1}{2}\theta\theta\bar\theta_{\dot\alpha}\Box\bar\chi^{\dot\alpha}(x) + i\theta^\alpha\sigma^\rho_{\alpha\dot\alpha}\Big(\partial_\rho\bar\chi^{\dot\alpha}(x) + \theta^\beta(\sigma^\mu\varepsilon)_\beta^{\dot\alpha}\partial_\rho w_\mu(x) + \bar\theta_{\dot\beta}\partial_\rho n^{\dot\beta\dot\alpha}(x)$$

$$+ \theta\sigma^\mu\bar\theta\partial_\rho\bar\xi^{\dot\alpha}_\mu(x) - \tfrac{i}{2}\overline{\theta\theta}\theta^\beta\sigma^\mu_{\beta\dot\beta}\partial_\mu\partial_\rho n^{\dot\beta\dot\alpha}(x)\Big)$$

$$= \varepsilon^{\alpha\beta}m_{\beta\alpha}(x) - \varepsilon_{\dot\alpha\dot\beta}n^{\dot\beta\dot\alpha}(x) + \theta(2\psi - \sigma^\mu\bar\xi_\mu(x) + i\sigma^\mu\partial_\mu\bar\chi(x)) + \bar\theta(4\bar\omega - \bar\sigma^\mu\lambda_\mu(x) - i\bar\sigma^\mu\partial_\mu\varphi(x))$$

$$+ 2i\theta\theta\partial^\mu w_\mu(x) - 2i\overline{\theta\theta}\partial^\mu v_\mu(x) + i\theta^\alpha\bar\theta_{\dot\beta}\Big(\bar\sigma^{\mu\dot\beta\beta}\partial_\mu(m_{\alpha\beta}(x) - m_{\beta\alpha}(x))$$

$$+ \sigma^\mu_{\alpha\dot\alpha}\partial_\mu(n^{\dot\beta\dot\alpha}(x) - n^{\dot\alpha\dot\beta}(x))\Big) + \tfrac{1}{2}\theta\theta\bar\theta(i\bar\sigma^\nu\sigma^\mu\partial_\mu\bar\xi_\nu(x) - \Box\bar\chi(x))$$

$$\tfrac{1}{2}\overline{\theta\theta}\theta(i\sigma^\nu\bar\sigma^\mu\partial_\mu\lambda_\nu(x) + \Box\varphi(x)) - \tfrac{1}{4}\theta\theta\overline{\theta\theta}(\varepsilon^{\alpha\beta}\Box m_{\beta\alpha}(x) - \varepsilon_{\dot\alpha\dot\beta}\Box n^{\dot\beta\dot\alpha}(x))$$

Separate $m_{\alpha\beta}(x)$, $n^{\dot\alpha\dot\beta}(x)$ into symmetric and antisymmetric parts:

$$m_{\alpha\beta}(x) = \tfrac{1}{2}\varepsilon_{\alpha\beta}m(x) + (\sigma^{\nu\mu}\varepsilon)_{\alpha\beta}m_{\mu\nu}(x), \tag{47a}$$

where $m(x) = \frac{1}{2}\varepsilon^{\beta\alpha}m_{\alpha\beta}(x)$, $\sigma^{\mu\nu} = \frac{1}{4}(\sigma^{\mu}\bar{\sigma}^{\nu}-\sigma^{\nu}\bar{\sigma}^{\mu})$ and $m_{\mu\nu}(x) = \frac{1}{2}(\varepsilon\sigma_{\mu\nu})^{\beta\alpha}m_{\alpha\beta}(x)$ is anti-symmetric and self-dual,

$$n^{\dot{\alpha}\dot{\beta}}(x) = \varepsilon^{\dot{\alpha}\dot{\beta}}n(x) + (\bar{\sigma}^{\nu\mu}\varepsilon)^{\dot{\alpha}\dot{\beta}}m_{\mu\nu}(x) \tag{47b}$$

where $n(x) = \frac{1}{2}\varepsilon_{\dot{\beta}\dot{\alpha}}n^{\dot{\alpha}\dot{\beta}}(x)$, $\bar{\sigma}^{\mu\nu} = \frac{1}{4}(\bar{\sigma}^{\mu}\sigma^{\nu}-\bar{\sigma}^{\nu}\sigma^{\mu})$ and $n_{\mu\nu}(x) = \frac{1}{2}(\varepsilon\bar{\sigma}_{\mu\nu})_{\dot{\beta}\dot{\alpha}}n^{\dot{\alpha}\dot{\beta}}(x)$ is antisymmetric and anti-self-dual. Then (38) implies that

$$m(x) - n(x) = 0 \tag{48a}$$

$$2\psi - \sigma^{\mu}\bar{\xi}_{\mu}(x) + i\sigma^{\mu}\partial_{\mu}\bar{\chi}(x) = 0 \tag{48b}$$

$$4\bar{\varpi} - \bar{\sigma}^{\mu}\lambda_{\mu}(x) - i\bar{\sigma}^{\mu}\partial_{\mu}\varphi(x) = 0 \tag{48c}$$

$$\partial^{\mu}w_{\mu}(x) = 0 \tag{48d}$$

$$\partial^{\mu}v_{\mu}(x) = 0 \tag{48e}$$

$$\partial_{\mu}m(x) + \partial_{\mu}n(x) = 0 \tag{48f}$$

$$i\bar{\sigma}^{\nu}\sigma^{\mu}\partial_{\mu}\bar{\xi}_{\nu}(x) - \Box\bar{\chi}(x) = 0 \tag{48g}$$

$$i\sigma^{\nu}\bar{\sigma}^{\mu}\partial_{\mu}\lambda_{\nu}(x) + \Box\varphi(x) = 0 \tag{48h}$$

$$\Box m(x) - \Box n(x) = 0 \tag{48i}$$

(47a) and (47f) give that $m(x)$ is a constant

$$m(x) = m \tag{48j}$$

and

$$n(x) = -m \,,$$

with (47i) following.

(47b) implies (47g) if $\partial^{\mu}\bar{\xi}_{\mu}(x) = 0$ is used, and (47c) implies (47h), using $\partial^{\mu}\lambda_{\mu}(x) = 0$.

Similarly

$$\bar{D}^{\dot{\alpha}}\Phi_{\alpha}(x,\theta,\bar{\theta}) + D_{\alpha}\bar{X}^{\dot{\alpha}}(x,\theta,\bar{\theta})$$

$$= (\bar{\sigma}^{\mu}\varepsilon)^{\dot{\alpha}}_{\alpha}v_{\mu}(x) + (\sigma^{\mu}\varepsilon)^{\dot{\alpha}}_{\alpha}w_{\mu}(x) + \theta^{\beta}\big(2\varepsilon_{\alpha\beta}\bar{\varpi}^{\dot{\alpha}} - (\sigma^{\mu}\varepsilon)^{\dot{\alpha}}_{\beta}\lambda_{\mu\alpha}(x) + i(\sigma^{\mu}\varepsilon)^{\dot{\alpha}}_{\beta}\partial_{\mu}\varphi_{\alpha}(x)\big)$$

$$+ \bar{\theta}_{\dot{\beta}}\big(\varepsilon^{\dot{\beta}\dot{\alpha}}\psi_{\alpha} + (\sigma^{\mu}\varepsilon)^{\dot{\beta}}_{\alpha}\bar{\xi}^{\dot{\alpha}}_{\mu}(x) + i(\sigma^{\mu}\varepsilon)^{\dot{\beta}}_{\alpha}\partial_{\mu}\bar{\chi}^{\dot{\alpha}}(x)\big) - i\theta\theta\bar{\sigma}^{\mu\dot{\alpha}\beta}\partial_{\mu}m_{\beta\alpha}(x) - i\bar{\theta}\bar{\theta}\sigma^{\mu}_{\alpha\dot{\beta}}\partial_{\mu}n^{\dot{\beta}\dot{\alpha}}(x)$$

$$+ \frac{1}{2}i\theta\sigma^{\rho}\bar{\theta}\big(((\bar{\sigma}^{\rho}\sigma^{\mu}\bar{\sigma}^{\nu}\varepsilon)^{\dot{\alpha}}_{\alpha} + (\bar{\sigma}^{\mu}\sigma^{\rho}\bar{\sigma}^{\nu}\varepsilon)^{\dot{\alpha}}_{\alpha})\partial_{\mu}v_{\nu}(x) - ((\bar{\sigma}^{\nu}\sigma^{\mu}\bar{\sigma}^{\rho}\varepsilon)^{\dot{\alpha}}_{\alpha} + (\bar{\sigma}^{\nu}\sigma^{\rho}\bar{\sigma}^{\mu}\varepsilon)^{\dot{\alpha}}_{\alpha})\partial_{\mu}w_{\nu}(x)\big)$$

$$+ \frac{1}{2}\theta\theta\bar{\theta}_{\dot{\beta}}(\varepsilon^{\dot{\alpha}\dot{\beta}}\Box\varphi_{\alpha}(x) - i(\bar{\sigma}^{\mu}\sigma^{\nu}\varepsilon)^{\dot{\alpha}\dot{\beta}}\partial_{\mu}\lambda_{\nu\alpha}(x)) + \frac{1}{2}\theta^{\beta}\bar{\theta}\bar{\theta}(\varepsilon_{\alpha\beta}\Box\bar{\chi}^{\dot{\alpha}}(x) - i(\sigma^{\nu}\bar{\sigma}^{\mu}\varepsilon)_{\beta\alpha}\partial_{\mu}\bar{\xi}^{\dot{\alpha}}_{\nu}(x))$$

$$\frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}((\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\rho}\varepsilon)^{\dot{\alpha}}_{\alpha}\partial_{\mu}\partial_{\nu}v_{\rho}(x) + (\sigma^{\mu}\bar{\sigma}^{\nu}\sigma^{\rho}\varepsilon)^{\dot{\alpha}}_{\alpha}\partial_{\mu}\partial_{\nu}w_{\rho}(x))$$

So that (38) implies that

$$w_{\mu}(x) = - v_{\mu}(x) \tag{49a}$$

$$i\partial_\mu\varphi_\alpha(x) - \lambda_{\mu\alpha}(x) + \sigma_{\mu\alpha\dot\alpha}\overline{\omega}^{\dot\alpha} = 0 \tag{49b}$$

$$i\partial_\mu\overline{\chi}^{\dot\alpha}(x) + \overline{\xi}^{\dot\alpha}_\mu(x) - \tfrac{1}{2}\overline{\sigma}_\mu^{\dot\alpha\alpha}\psi_\alpha = 0 \tag{49c}$$

$$\overline{\sigma}^{\mu\dot\alpha\beta}\partial_\mu m_{\beta\alpha}(x) = 0 \tag{49d}$$

$$\sigma^\mu_{\alpha\dot\beta}\partial_\mu n^{\dot\beta\dot\alpha}(x) = 0 \tag{49e}$$

$$(\overline{\sigma}^\rho\sigma^\mu\overline{\sigma}^\nu + \overline{\sigma}^\mu\sigma^\rho\overline{\sigma}^\nu)\partial_\mu v_\nu(x) - (\overline{\sigma}^\nu\sigma^\mu\overline{\sigma}^\rho + \overline{\sigma}^\nu\sigma^\rho\overline{\sigma}^\mu)\partial_\mu w_\nu(x) = 0 \tag{49f}$$

$$\varepsilon^{\dot\alpha\dot\beta}\Box\varphi_\alpha(x) - i(\overline{\sigma}^\mu\sigma^\nu\varepsilon)^{\dot\alpha\dot\beta}\partial_\mu\lambda_{\nu\alpha}(x) = 0 \tag{49g}$$

$$\varepsilon_{\alpha\beta}\Box\overline{\chi}^{\dot\alpha}(x) - i(\sigma^\nu\overline{\sigma}^\mu\varepsilon)_{\beta\alpha}\partial_\mu\overline{\xi}^{\dot\alpha}_\nu(x) = 0 \tag{49h}$$

$$\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho\partial_\mu\partial_\nu v_\rho(x) + \sigma^\mu\overline{\sigma}^\nu\sigma^\rho\partial_\mu\partial_\nu w_\rho(x) = 0 \tag{49i}$$

(49a) in (49f) implies that

$$2(\eta^{\rho\mu}\overline{\sigma}^\nu + \eta^{\mu\nu}\overline{\sigma}^\rho - \eta^{\rho\nu}\overline{\sigma}^\mu + \eta^{\rho\mu}\overline{\sigma}^\nu + \eta^{\rho\nu}\overline{\sigma}^\mu - \eta^{\mu\nu}\overline{\sigma}^\rho)\partial_\mu v_\nu(x) = 0$$

i.e. that

$$\partial_\mu v_\nu(x) = 0 \tag{50}$$

so that v_μ and $w_\mu = - v_\mu$ are constant.
(48b) and (49c) give

$$i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\overline{\chi}^{\dot\alpha}(x) = 0 \tag{51}$$

and

$$\overline{\xi}^{\dot\alpha}_\mu(x) = -i\partial_\mu\overline{\chi}^{\dot\alpha}(x) + \tfrac{1}{2}\overline{\sigma}^{\mu\dot\alpha\alpha}\psi_\alpha$$

(48c) and (49b) that

$$i\overline{\sigma}^{\mu\dot\alpha\alpha}\partial_\mu\varphi_\alpha(x) = 0 \tag{52}$$

and

$$\lambda_{\mu\alpha}(x) = i\partial_\mu\varphi_\alpha(x) + \sigma_{\mu\alpha\dot\alpha}\overline{\omega}^{\dot\alpha}$$

(49d), (47a) and (48j) give

$$\overline{\sigma}^\rho\sigma^{\nu\mu}\partial_\rho m_{\mu\nu}(x) = 0$$

Using

$$(\sigma^\sigma\overline{\sigma}^\rho)_\alpha^{\ \beta} = \tfrac{1}{2}(\sigma^{\sigma\rho})_\alpha^{\ \beta} + \eta^{\sigma\rho}\delta_\alpha^\beta$$

$$\mathrm{tr}(\sigma^{\mu\nu}) = 0$$

$$\mathrm{tr}(\sigma^{\sigma\rho}\sigma^{\mu\nu}) = -\tfrac{1}{2}(\eta^{\sigma\mu}\eta^{\rho\nu} - \eta^{\sigma\nu}\eta^{\rho\mu} + i\varepsilon^{\sigma\rho\mu\nu})$$

and the self-dual, anti-symmetric nature of $m_{\mu\nu}(x)$,

$$\partial_\rho m^{\sigma\rho}(x) = 0 \tag{53}$$

Similarly (49e), (47b) and (48j) give

$$\partial^{\mu} n_{\mu\nu}(x) = 0 \tag{54}$$

Define

$$A_{\mu\nu}(x) = m_{\mu\nu}(x) + n_{\mu\nu}(x)\,, \tag{55}$$

then (53) and (54) imply that

$$\partial^{\mu} A_{\mu\nu}(x) = 0 \tag{56}$$

and also that

$$\begin{aligned} \tfrac{i}{2}\varepsilon^{\sigma\rho\mu\nu}\partial_{\rho} A_{\mu\nu}(x) &= \partial_{\rho} m^{\sigma\rho}(x) - \partial_{\rho} n^{\sigma\rho}(x) \\ &= 0 \end{aligned}$$

The latter implies that $A_{\mu\nu}(x)$ is the curl of a vector, $A_{\mu}(x)$ say,

$$A_{\mu\nu}(x) = \partial_{\mu} A_{\nu}(x) - \partial_{\nu} A_{\mu}(x) \tag{56}$$

This exhausts the content of the massless superfield equation (38).

Summarizing, we have found that the only propagating fields in the massless square root of the Dirac equation are a Dirac spinor

$$\psi(x) = \begin{pmatrix} \varphi_{\alpha}(x) \\ \overline{\chi}^{\dot{\alpha}}(x) \end{pmatrix}$$

satisfying the Dirac equation

$$i\partial\!\!\!/\psi(x) = 0$$

and a complex vector $A_{\mu}(x)$ appearing gauge invariantly through

$$A_{\mu\nu}(x) = \partial_{\mu} A_{\nu}(x) - \partial_{\nu} A(x)$$

and satisfying the equation of motion

$$\partial^{\mu} A_{\mu\nu}(x) = 0$$

Note that if the restriction to a Majorana-spinor superfield is made, then the field content is the same as that of the vector multiplet. Indeed the formulation (13) of the supersymmetric gauge theory gives a solution of (38); not of the constraints (23), however. A treatment which made more use of the superfield formulation, rather than the component field one given above, might reveal if, in fact, the two models are equivalent.

There are a number of other aspects of the square root of the Dirac equation which might be explored. A treatment of the bosonic superfield would enable the massive case to be examined. Interaction terms which might be added to the free action

$$\int d^4x\, d^2\theta\, d^2\overline{\theta}\, \Big(\langle B(x,\theta,\overline{\theta}), A\Psi(x,\theta,\overline{\theta})\rangle + \Lambda^{\alpha}(x,\theta,\overline{\theta}) M(\tfrac{1}{4},\tfrac{1}{2})\Phi_{\alpha}(x,\theta,\overline{\theta}) + \overline{\Lambda}_{\dot{\alpha}}(x,\theta,\overline{\theta})\, M(\tfrac{1}{2},\tfrac{1}{4})\overline{X}^{\dot{\alpha}}(x,\theta,\overline{\theta})\Big)$$

should also be considered, including the coupling to supergravity.

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Chapter 3
Supersymmetric Quantum Mechanics and the Index Theorem

The Atiyah-Singer index theorem [1] is a result in differential geometry which was established in the sixties. Although the general result and its standard proof, which involves K-theory, are beyond most theoretical physicists, the statement of the theorem for specific cases is certainly not. One such case is the twisted Dirac operator or, in the language of theoretical physics, the Dirac operator in the presence of gravitational and gauge fields. In the seventies and eighties this case, together with some of its generalizations, was found to be of use in theoretical physics: primarily in the study of anomalies [2], but also in the analysis of the fermion spectrum in Kaluza-Klein theories [3]. At the same time an index for supersymmetric quantum theories was defined [4] - the Witten index. It was a short step to see the index of the Dirac operator as the Witten index of an appropriate supersymmetric system, and then to attempt to arrive at the index theorem result by calculating the Witten index using the methods of theoretical physics [5]-[8]. Of course the proof that results does not have the rigour of the mathematicians' proof but it is comprehensible to theoretical physicists.

In the first part of this chapter, we will review the Atiyah-Singer index theorem and give its statement in a number of cases - in particular, in the case of the twisted Dirac operator. We will then go on to show how this leads easily to the gravitational index therems for fields of arbitrary spin [9] and calculate the general expression in four dimensions. Finally, we will briefly review the applications of the index theorem to anomales and to Kaluza-Klein theories.

In the second part of this chapter we will carefully derive the statement of the index theorem for the Dirac and twisted Dirac cases via the Witten index and using elementary path integral methods. While this has been done before [5]-[7], we will consider certain aspects in the construction of the supersymmetric quantum mechanics and in the construction and evaluation of the path integral at greater length. In particular, this will ensure that the normalization of the final result is properly determined through the calculation and not imposed at the end, and also that any other ambiguities which arise are properly dealt with. In the process we hope to have thrown some light on the path integrals themselves.

I. The Atiyah-Singer Index Theorem

1. Introduction

The basic statement of the Atiyah-Singer index theorem is [1],[10] :
Let M be a compact, smooth, n-dimensional manifold; let D be an elliptic operator on M, $D:C^{\infty}(E)\to C^{\infty}(F)$, where E and F are complex vector bundles over M; and let $\sigma(D)$ be the symbol bundle of D. Then

$$\text{index } D = (-1)^{n}\text{ch}(\sigma(D))\text{td}(TM\otimes C)[TM]$$

For the purposes of explanation it is easier to write this as

$$\text{index } D = (-1)^{\frac{1}{2}n(n+1)}\frac{\text{ch}(E-F)\text{td}(TM\otimes C)}{e(TM)}[M]$$

although to understand this in more than a formal manner is not straightforward either.

Let us first consider what is meant by a vector bundle on M. An m-dimensional vector bundle V over (or with base manifold) M is a manifold which seems locally to be a Cartesian product of M with a vector space, the fibre, isomorphic to R^m if V is real or to C^m if Vis complex, but which globally may be in some way twisted through the linear action of a group, G, on the fibre. For physics the most important examples are the bundles with G a Yang-Mills gauge group and, in the study of gravitation, the tangent bundle of M, TM, which has fibre R^n and group Gl(n,R), together with the bundles derived from it: the complexified tangent bundle $TM\otimes C$, with fibre C^n; the cotangent bundle T^*M and the tensor bundles on M; the bundle of k-forms, $\Lambda^k(M)$, k=0,..,n ; and the spinor bundle, $\Delta(M)$, which if it exists can be constructed from TM together with a Riemannian metric, it has fibre $C^{[n/2]}$ and group Spin(n). These bundles are constructed from TM by vector space operations (tensor products, sums, Grassmann and Clifford multiplication) on the fibre, this can be done in general for vector bundles over a common base manifold. In fact it is even possible, in the context of K-theory and characterisitic classes, to extend these operations, hence the occurrence of a difference of two vector bundles, E-F in the index formula.

A section of a vector bundle, V, over M is a mapping of a neighbourhood, U, of M into V such that the image of a point p in U is a point s(p) in V "above" p. Using the local product structure of V, s can be written locally as $s:p\to(p,u(p))$, where $u(p)\in R^m$ or C^m - that is a section corresponds to the notion of a vector field. The space of smooth sections of V is denoted $C^{\infty}(V)$. Linear differential operators are defined between smooth sections of vector bundles. Locally a linear differential operator $D:C^{\infty}(E)\to C^{\infty}(F)$ can be written as $D:(p,u(p))\to(p,(Du)(p))$,

$$(Du)(p) = \sum_{|k|=0}^{m} a_k(p)(D^k u)(p)$$

where $k=(k_1,\ldots,k_n)$ is a multi-index, $|k| = k_1+\ldots+k_n$, $a_k(p)$ is a linear transformation between the fibres of E and F, and, given x^{μ} as co-ordinates for M,

$$D^k = \left(\frac{\partial}{\partial x^1}\right)^{k_1} \cdots \left(\frac{\partial}{\partial x^n}\right)^{k_n}.$$

The index theorem requires that the operator D be elliptic, this means that the principal symbol of D,

$$\sigma_p(\xi) = \sum_{|k|=m} a_k(p)\xi^k,$$

where $\xi \in R^n$ (or $T_p{}^*(M)$) and $\xi^k = \xi_1{}^{k_1} \ldots \xi_n{}^{k_n}$, must be an isomorphism between the fibres of E and F for all $p \in M$, $\xi \neq 0$. Obviously then, E and F must have the same dimension.

We are now in a position to give the definition of the index of D. It is

$$\text{index } D = \dim(\ker D) - \dim(\operatorname{coker} D).$$

For an elliptic operator the kernel and the cokernel must be finite dimensional and so the index is well-defined. The index is also then given by

$$\dim(\ker D) - \dim(\operatorname{coker} D^*),$$

where D^* is the dual of D with respect to some inner products on $C^\infty(E)$ and $C^\infty(F)$.

The right hand side of the index formula (2) consists of characteristic classes of vector bundles, with the whole evaluated on the base manifold M. Characteristic classes are objects in algebraic topology, in K-theory or cohomology. K-theory is the more suitable algebraic topology for consideration of the theorem itself, however, in the particular examples of interest it is through cohomolgy that the physical content becomes plain. A characteristic class is defined either for complex or for real vector bundles; of the characteristic classes occurring inthe index formula (2) the Chern character, ch, and the Todd class, td, are complex and the Euler class, e, is real. In the examples below these will reduce to other classes which will then be considered more explicitly through cohomology in terms of invariant polynomials in the curvartures of the vector bundles involved. The evaluation on M then just becomes an integral over M of the n-form part.

The concepts of connection and curvature for vector bundles are more general than the like in theoretical physics. However, for the bundles derived from a tangent bundle with Riemannian metric, the connection and curvature can be brought into coincidence with those of gravitation, and for a vector bundle with a Yang-Mills type group with the Yang-Mills ones. That is the connection and curvature can be thought of locally as one- and two-forms respectively on M taking values in the Lie algebra of G and with certain transformation properties giving their global structure.

Although the characteristic classes will be expressed in terms of the curvature, it is fundamental that they are concerned only with the vector bundle involved, and that they are quite independent not only of the choice of gauge or co-ordinate system but even of the connection used to define the curvature involved. That is they will change only by a total derivative, which when integrated over M, which is compact and without boundary, will vanish.

Summing up then, the left hand side of the index theorem formula involves the solutions to differential equations over M, while the right hand side involves only global topological quantities, and in fact, as can be seen from (2), the operator D only enters the expression through E and F, that is through the spaces between which it acts.

There are a number of generalizations of the Atiyah-Singer index theorem [12]. For compact manifolds with boundary and non-compact manifolds the Atiyah-Patodi-Singer index theorem gives the necessary corrections to the ordinary index theorem. Then there is the G-index theorem or character-valued index theorem. When there is a group G, with an action on M and a linear action on E and F which commutes with D, the g-index of D for $g \in G$ is

$$\mathrm{index}_g D = \mathrm{tr}_{\ker D}(g) - \mathrm{tr}_{\mathrm{coker} D}(g).$$

The G-index theorem gives a topological expression for this index. Finally, there is the family's index theorem, giving a topological expression for the index of a family of differential operators, a set of differential operators which are themselves parametrized by some compact manifold.

2. The Index Theorem for the Exterior Derivative of Differential Forms

Let M be an oriented manifold with even dimension, n=2l. The bundle of differential forms over M,

$$\Lambda^*(M) = \bigoplus_{p=0}^{n} \Lambda^p(M),$$

consists of the completely anti-symmetric covariant tensors at points of M. The exterior derivative operator d acts on the sections of $\Lambda^*(M)$, which we also call forms for convenience. Denoting by d_p the exterior derivative restricted to forms of rank p,

$$d_p: C^\infty(\Lambda^p(M)) \to C^\infty(\Lambda^{p+1}(M)).$$

If M possesses a Riemannian metric, then the Hodge star operator, *, can be formed,

$$*_p: \Lambda^p(M) \to \Lambda^{n-p}(M), \quad *_{n-p} *_p = (-1)^{p^2}$$

and the inner product on $C^\infty(\Lambda^p(M))$, p=0,...,n,

$$\langle \alpha, \beta \rangle_p = \int_M \alpha \wedge * \beta \,.$$

The dual of d_p with respect to $\langle\ ,\ \rangle_p$, $\langle\ ,\ \rangle_{p+1}$ is denoted δ_{p+1} and is given by

$$\delta_{p+1}: C^\infty(\Lambda^{p+1}(M)) \to C^\infty(\Lambda^p(M))$$

$$\delta = -*d*.$$

The operator $d+\delta$ on $C^\infty(\Lambda^*(M))$ is elliptic and complexifying $\Lambda^*(M)$, denoting its complexification $\Lambda^*(M) \otimes C$ by

$$A^* = \bigoplus_{p=0}^{n} A^p$$

and extending d, δ and* to A^* we may consider the index theorem applied to the operator

$$d+\delta: C^\infty(A^*)\to C^\infty(A^*).$$

However, this is uninteresting as d+δ is self-dual and so

$$\mathrm{ind}(d+\delta) = \dim(\ker(d+\delta)) - \dim(\ker(d+\delta)^*)$$

is trivially zero, as is the other side of the index theorem formula.

In order to achieve an interesting result we must split A^* into two bundles of equal dimension, $A^* = E\oplus F$, in such a way that d+δ is off-diagonal

$$(d+\delta)_E: C^\infty(E)\to C^\infty(F)\,,\quad (d+\delta)_E^* = (d+\delta)_F$$

Then

$$\mathrm{ind}((d+\delta)_E) = \dim(\ker((d+\delta)_E)) - \dim(\ker(d+\delta)_F)).$$

One way in which this may be done is by splitting A^* into forms of odd and even rank.

$$A^* = A^{odd}\oplus A^{even} = \bigoplus_{i=0}^{l} A^{2i} \oplus \bigoplus_{i=0}^{l-1} A^{2i-1}$$

Then the index theorem result

$$\mathrm{ind}((d+\delta)_{even}) = \int_M \frac{ch(A^{even}-A^{odd})td(TM\otimes C)}{e(TM)}$$

can be reduced to

$$\mathrm{ind}((d+\delta)_{even}) = \int_M e(TM) \qquad (3)$$

where once again e is the Euler class. In terms of the curvature two-form on M, $R_{\alpha\beta}$, taking values in SO(n)

$$e = \frac{1}{(4\pi)^l l!}\varepsilon^{\alpha_1\ldots\alpha_n}R_{\alpha_1\alpha_2}\wedge\ldots\wedge R_{\alpha_{n-1}\alpha_n}\,.$$

The result (3) also holds for manifolds of odd dimension, but then A^{odd} and A^{even} are isomorphic under * and the index vanishes, the Euler class is also identically zero.

Since d+δ and the splitting $A^* = A^{even}\oplus A^{odd}$ are real, the index of d+δ on $C^\infty(A^{even})$ is just the index of d+δ on $C^\infty(A^{even}(M))$. Note also that

$$\ker\delta_p = C^\infty(\Lambda^p(M))\backslash \mathrm{im}\, d_{p-1} \quad \text{and} \quad \mathrm{im}\, d_{p-1}\subseteq \ker d_p$$

so that

$$\begin{aligned}\ker(d_p+\delta_p) &= \ker d_p \cap \ker\delta_p\\ &\approx \ker d_p\backslash \mathrm{im}\, d_{p-1}\,.\end{aligned}$$

and

$$\begin{aligned}\mathrm{ind}((d+\delta)_{even}) &= \dim(\ker(d+\delta)_{even}) - \dim(\ker(d+\delta)_{odd})\\ &= \sum_p (-1)^p \dim(\ker(d+\delta)_p)\\ &= \sum_p (-1)^p \dim(\ker d_p/\mathrm{im}\, d_{p-1})\,,\end{aligned}$$

which is the index of the elliptic complex

$$0 \to C^\infty(\Lambda^0(M)) \to \ldots \to C^\infty(\Lambda^{p-1}(M)) \xrightarrow{d_{p-1}} C^\infty(\Lambda^p(M)) \xrightarrow{d_p} \ldots \to C^\infty(\Lambda^n(M)) \to 0$$

It is only a slight generalization to write the index theorem in terms of elliptic complexes.

Now $\ker d_p/\mathrm{im}\, d_{p-1}$ is the pth de Rham cohomology group of M, $H_{DR}{}^p(m;R)$, and is by de Rham's theorem isomorphic to the singular cohomology group of M, $H^p(M;R)$.

$$\sum_p (-1)^p \dim H^p(M;R)$$

is the Euler characteristic of M, $\chi(M)$. Thus, in this case, the index theorem is just the Gauss-Bonnet theorem

$$\chi(M) = \int_M e(TM)$$

Another splitting of A^* is into self-dual and anti-self-dual forms. That is setting

$$\omega_p = i^{p(p-1)+1} *_p$$

so that $\omega^2 = 1$ and $\omega(d+\delta) = -(d+\delta)$, split A^* into the +1 and -1 eigenspaces of ω,

$$A^* = A^+ \oplus A^-$$

The index theorem result, in this case,

$$\mathrm{ind}\,((d+\delta)_+) = \int_M \frac{ch(A^+ - A^-) td(TM \otimes C)}{e(TM)}$$

reduces to

$$\mathrm{ind}((d+\delta)_+) = \int_M L(TM)\,, \tag{4}$$

where L is the Hirzebruch L-polynomial. It is given in terms of the curvature as

$$\det^{\frac{1}{2}}\left(\frac{\frac{iR}{2\pi}}{\tanh\frac{iR}{2\pi}}\right),$$

which is to be interpreted as a power series in $iR/2\pi$.

$$\det^{\frac{1}{2}}\left(\frac{\frac{iR}{2\pi}}{\tanh\frac{iR}{2\pi}}\right) = 1 - \frac{1}{24\pi^2}\,\mathrm{tr}(R \wedge R) + \ldots$$

containing only forms of degree divisible by four.

Using the fact that $\omega_p: A^p \to A^{n-p}$, it can be shown that the contribution to $\mathrm{ind}((d+\delta)_+)$ of forms of rank other than l cancel, and so

$$\mathrm{ind}((d+\delta)_+) = \dim(\ker((d+\delta)_{l,+})) - \dim(\ker(d+\delta)_{l,-})\,.$$

Furthermore, if l is odd then ω_l is pure imaginary and the index vanishes. If on the other hand l is even, then $\omega_l = *_l$ and we need consider only real forms once again. The index is just the signature of $\langle\ ,*\ \rangle$ on $\ker(d+\delta)_l$, or the topological signature of

M, $\tau(M)$. Thus the index theorem reduces to the Hirzebruch signature theorem

$$\tau(M) = \int_M L(M) .$$

3. The Index Theorem for the Dirac Operator

Our last example is of most relevance to physics and in a sense the most fundamental. Let M be a Riemannian manifold which admits a spin structure and let $\Delta(M)$ be the spinor bundle over M associated with the Riemannian metric on TM. Further, let V be another vector bundle over M with group G. Then $\Delta\otimes V$, the twisted spinor bundle, is a vector bundle over M whose sections are spinor fields on M carrying a representation of G.

Given a connection on V and the spin connection associated with the Riemannian metric on TM, the Dirac operator is the map

$$i\nabla\!\!\!/_V : C^\infty(\Delta\otimes V) \to C^\infty(\Delta\otimes V)$$

given locally in terms of an orthonormal basis $\{E_\alpha\}$ of TM , $g(E_\alpha,E_\beta) = \delta_{\alpha\beta}$, as

$$i\nabla\!\!\!/_V\psi = E_\alpha i\nabla_{E_\alpha}\psi\ ,\ \text{for}\ \psi\in C^\infty(\Delta\otimes V),$$

where ∇_{E_α} is the covariant derivative in the E_α direction and the product is Clifford multiplication. That is,

$$i\nabla\!\!\!/_V\psi = \gamma^\alpha E_\alpha^\mu i\nabla_\mu\psi$$

where $E_\alpha{}^\mu$ are the components of E_α in a co-ordinate system $\{x^\mu\}$, ∇_μ is the covariant derivative in the $\partial/\partial x^\mu$ direction, γ^α are the Dirac matrices, $\{\gamma^\alpha,\gamma^\beta\} = 2\delta^{\alpha\beta}$, and ψ is a Dirac spinor in an appropriate representation.

It can be seen that $i\nabla\!\!\!/_V$ is elliptic, as its leading symbol

$$\sigma_p(x,\xi)\psi = \gamma^\alpha E_\alpha^\mu i\xi_\mu\psi$$

has square

$$\sigma_p^2(x,\xi)\psi = -g(\xi,\xi)\psi = -g^{\mu\nu}\xi_\mu\xi_\nu\psi\ ,$$

which is just multiplication by a non-zero scalar for $\xi\neq 0$, since g is Riemannian (positive definite).

The Dirac operator is self-dual on the full space $C^\infty(\Delta\otimes V)$ and thus its index trivially zero, so once again the space must be split. If the dimension of M is even, n=2l say, then there is an element of the Clifford algebra the chirality operator

$$\Gamma = i^l\gamma^1\ldots\gamma^n$$

such that

$$\Gamma^2 = 1 \ \text{ and }\ \{\gamma^\alpha,\Gamma\} = 0,$$

and such that splitting $\Delta(M)$ into eigenspaces of Γ,

$$\Delta(M) = \Delta^+(M)\oplus\Delta^-(M)\ ,$$

$i\nabla\!\!\!/_V$ is off diagonal,

$$i\nabla\!\!\!/_{V+}: C^\infty(\Delta^+\otimes V) \to C^\infty(\Delta^-\otimes V)$$

with

$$(i\nabla\!\!\!/_{V+})^* = i\nabla\!\!\!/_{V-}\,.$$

The index theorem result here reduces to

$$\mathrm{ind}(i\nabla\!\!\!/_{V+}) = \int_M \hat{A}(M)\mathrm{ch}(V) \tag{5}$$

where A is the Dirac or A-roof genus

$$\hat{A}(M) = \det^{\frac{1}{2}}\left(\frac{\frac{iR}{4\pi}}{\sinh\frac{iR}{4\pi}}\right)$$

$$= 1 + \frac{1}{12(4\pi)^2}\,\mathrm{tr}(R\wedge R) + \ldots$$

and ch is the Chern character again, given in terms of the curvature Ω of V, which is a two-form on M taking values in the Lie algebra of G, as

$$\mathrm{ch}(V) = \mathrm{tr}\left(\exp\left(\frac{i\Omega}{2\pi}\right)\right).$$

This example of the Dirac operator on the twisted spinor bundle is fundamental in as much as the choice of the vector bundle V enables us to consider different spaces between which the operator is acting. In particular, by taking V to be some bundle derived from the tangent bundle and with the connection associated with the Riemannian metric we may find the index of an operator acting on a field carrying other than the spinor representation of SO(n) - i.e. of spin other than a half. Remembering that by the index theorem <u>the index depends only upon the spaces and not on the operator</u>, the index found will be the same as that of the appropriate physical operator. V can also be chosen to be a vector bundle with G a Yang-Mills group and Ω the Yang-Mills curvature, or a tensor product of such a bundle with one as above, giving the index for an operator acting on spinors, or on higher-spin fields, carrying a representation of the Yang-Mills gauge group.

4. Gravitational Index Theorems for Arbitrary Spin Fields

Christensen and Duff [13] have considered the problem of index theorems for fields of arbitrary spin in four dimensions. Their method was to construct an appropriate physical operator and then to calculate its index by the heat kernel method. They met with difficulties in the shape of consistency conditions which imposed restrictions upon the underlying manifold and which ultimately, for higher spins , meant that the topological invariants and thus the index had to vanish. We give below a simpler derivation [9] based on the twisted Dirac operator and which encounters no such restrictions.

The Riemannian metric can be used to restrict the group of TM from Gl(n,R) to SO(n). This is usually done in terms of the frame bundle, whose points are the bases of the tangent spaces at the points of M - it is an example of a principal fibre bundle, one whose fibre is homeomorphic to its group. The restriction to orthonormal bases with respect to the Riemannian metric then reduces the fibre and group to S. The resultant principal bundle is called the orthonormal frame bundle - we denote it O(M). For every representation,φ , of SO(n) a vector bundle can be associated with the orthonormal frame bundle, just the vector bundle whose fibre is the vector space carrying the representation and which is twisted in the same manner as O(M) but through the given representation.

Let us take V to be the vector bundle associated with O(M) by the representation φ. Let $J_{\alpha\beta}$ be the antihermitian generators of φ,

$$[J_{\alpha\beta},J_{\gamma\delta}] = \eta_{\alpha\delta}J_{\beta\gamma} + \eta_{\beta\gamma}J_{\alpha\delta} - \eta_{\alpha\gamma}J_{\beta\delta} - \eta_{\beta\delta}J_{\alpha\gamma}\,. \tag{6}$$

Then

$$\Omega = \tfrac{1}{2}R^{\alpha\beta}J_{\alpha\beta}$$

is the curvature of V, where

$$R^{\alpha\beta} = R^{\alpha\beta}_{\ \ \mu\nu}\, dx^{\mu}\wedge dx^{\nu}$$

is the curvature of M.

The index of $i\nabla_+$ is

$$I[\varphi] = \int_M \det^{\frac{1}{2}}\left(\frac{\sinh\frac{iR}{4\pi}}{\frac{iR}{4\pi}}\right)\mathrm{tr}\left(\exp\left(\frac{i}{4\pi}R^{\alpha\beta}J_{\alpha\beta}\right)\right) \tag{7}$$

To evaluate this explicitly we must take the terms of order l (n=2l) in R, evaluating the trace of products of up to l $J_{\alpha\beta}$'s.

Specializing to n=4, we have SO(4) = SO(3)$\otimes$SO(3), and so the reporesentations of SO(4) are just the direct product of two representations of SO(3). The finite-dimensional irreducible representations of SO(3)$\approx$SU(2) are of course just labelled by the non-negative half-integers. The representation labelled by s has dimension 2s+1. If L_a, a=1,2,3 , are anti-hermitian generators of s,

$$[L_a,L_b] = \varepsilon_{abc}L_c\,,$$

then

$$\mathrm{tr}(L_aL_b) = -\tfrac{1}{3}s(s+1)(2s+1)\delta_{ab}\,.$$

This can be seen by explicit construction of the L_a, or just by noting that in the adjoint representation

$$\mathrm{tr}(L_aL_b) = -2\delta_{ab}$$

and for arbitrary s the coefficient is just

$$\tfrac{1}{3}\mathrm{tr}(\underline{L}^2) = \tfrac{1}{3}\mathrm{tr}(-s(s+1)\mathbb{1}) = -\tfrac{1}{3}s(s+1)(2s+1).$$

A finite-dimensional, irreducible representation, φ, of SO(4) with antihermitian generators $J_{\alpha\beta}$ satisfying (6) can be decomposed into a product of two finite-dimensional, irreducible representations of SO(3) by

$$M_a = \frac{1}{4}\varepsilon_{abc}J_{bc} + \frac{1}{2}J_{a4}$$

$$N_a = \frac{1}{4}\varepsilon_{abc}J_{bc} - \frac{1}{2}J_{a4} ,$$

then

$$[M_a,M_b] = \varepsilon_{abc}M_c , \quad [N_a,N_b] = \varepsilon_{abc}N_c .$$

If the M_a and the N_a generate the $2s_1+1$ and $2s_2+1$ dimensional representations of SO(3) respectively. Then φ denoted (s_1,s_2) is of dimension $(2s_1+1)(2s_2+1)$,

$$\begin{aligned} tr(J_{ab}J_{cd}) &= \varepsilon_{abe}\varepsilon_{cdf}tr((M_e+N_e)(M_f+N_f)) \\ &= \varepsilon_{abe}\varepsilon_{cdf}(tr_{2s_1+1}(M_eM_f)tr_{2s_2+1}(\mathbb{1}) + tr_{2s_1+1}(\mathbb{1})tr_{2s_2+1}(N_eN_f)) \\ &= \frac{1}{3}(\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd})(2s_1+1)(2s_2+1)(s_1(s_1+1) + s_2(s_2+1)), \\ tr(J_{ab}J_{c4}) &= \varepsilon_{abd}tr((M_d + N_d)(M_c - N_c)) \\ &= -\frac{1}{3}\varepsilon_{abc}(2s_1+1)(2s_2+1)(s_1(s_1+1) - s_2(s_2+1)) \end{aligned}$$

and

$$\begin{aligned} tr(J_{a4}J_{b4}) &= tr((M_a - N_a)(M_b - N_b)) \\ &= -\frac{1}{3}\delta_{ab}(2s_1+1)(2s_2+1)(s_1(s_1+1) + s_2(s_2+1)) , \end{aligned}$$

i.e

$$\begin{aligned} tr(J_{\alpha\beta}J_{\gamma\delta}) = {} & \frac{1}{3}(\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta})(2s_1+1)(2s_2+1)(s_1(s_1+1) + s_2(s_2+1)) \\ & - \frac{1}{3}\varepsilon_{\alpha\beta\gamma\delta}(2s_1+1)(2s_2+1)(s_1(s_1+1) - s_2(s_2+1)) . \qquad (8) \end{aligned}$$

Let us denote by $\Delta(s_1,s_2)$ the vector bundle associated with O(M) by the representation (s_1,s_2). Then $\Delta^+(M) = \Delta(1/2,0)$, $\Delta^-(M) = \Delta(0,1/2)$ and TM with group restricted to SO(4) is $\Delta(1/2,1/2)$. If V is $\Delta(s_1,s_2)$, then the twisted Dirac operator is an elliptic operator between the sections of the bundles

$$\Delta(\tfrac{1}{2},0)\otimes\Delta(s_1,s_2) = \Delta(s_1+\tfrac{1}{2},s_2)\oplus\Delta(s_1-\tfrac{1}{2},s_2)$$

and

$$\Delta(0,\tfrac{1}{2})\otimes\Delta(s_1,s_2) = \Delta(s_1,s_2+\tfrac{1}{2})\oplus\Delta(s_1,s_2-\tfrac{1}{2}) ,$$

where the decomposition into direct sums follow from the rules for products of representations of SO(3) and the second term is omitted when s_1 (or s_2) is zero.

The index of $i\nabla\!\!\!/_{V+}$ is

$$I[s_1,s_2] = \int_M \left(1 + \frac{1}{12(4\pi)^2}\,\mathrm{tr}(R\wedge R)\right)\left(\mathrm{tr}(\mathbb{1}) + \frac{1}{2!}\left(\frac{i}{4\pi}\right)^2 R^{\alpha\beta}\wedge R^{\gamma\delta}\,\mathrm{tr}(J_{\alpha\beta}J_{\gamma\delta})\right)$$

$$= \frac{1}{(4\pi)^2}(2s_1+1)(2s_2+1)\int_M \Big(\frac{1}{12}\mathrm{tr}(R\wedge R) - \frac{1}{3}\mathrm{tr}(R\wedge R)(s_1(s_1+1) + s_2(s_2+1))$$
$$+ \frac{1}{6}\varepsilon_{\alpha\beta\gamma\delta}R^{\alpha\beta}\wedge R^{\gamma\delta}(s_1(s_1+1) - s_2(s_2+1))\Big)$$

$$= \frac{1}{6}(2s_1+1)(2s_2+1)(-\frac{1}{4} + s_1(s_1+1) + s_2(s_2+1))P(M)$$
$$+ \frac{1}{3}(2s_1+1)(2s_2+1)(s_1(s_1+1) - s_2(s_2+1))\chi(M) \qquad (9)$$

where

$$P(M) = -\frac{2}{(4\pi)^2}\int_M R^{\alpha\beta}\wedge R_{\beta\alpha}$$

is the Pontrjagin number of M, and

$$\chi(M) = \frac{1}{2(4\pi)^2}\int_M \varepsilon_{\alpha\beta\gamma\delta}R^{\alpha\beta}\wedge R^{\gamma\delta}$$

is the Euler number of M. This then is the general result for the index of an elliptic operator between fields differing by integral spin.

The first few cases are:

$$I[0,0] = -\frac{P}{24}$$

the index of the ordinary Dirac operator, related to the spin-1/2 axial anomaly (see below);

$$I[\tfrac{1}{2},0] + I[0,\tfrac{1}{2}] = \frac{P}{3}$$

the index of an operator

$$C^\infty(\Delta(1,0)\oplus\Delta(0,0)\oplus\Delta(\tfrac{1}{2},\tfrac{1}{2})) \to C^\infty(\Delta(0,1)\oplus\Delta(0,0)\oplus\Delta(\tfrac{1}{2},\tfrac{1}{2}))\ ,$$

which is the Hirzebruch signature theorem;

$$I[\tfrac{1}{2},0] - I[0,\tfrac{1}{2}] = \chi\ ,$$

the index of an operator

$$C^\infty(\Delta(1,0)\oplus\Delta(0,1)\oplus 2\Delta(0,0)) \to C^\infty(2\Delta(\tfrac{1}{2},\tfrac{1}{2}))\ ,$$

which is the Gauss-Bonnet theorem; and

$$I[\tfrac{1}{2},\tfrac{1}{2}] - I[0,0] = \frac{21P}{24}\ ,$$

the index of an operator

$$C^\infty(\Delta(1,\tfrac{1}{2})) \to C^\infty(\Delta(\tfrac{1}{2},1))$$

which is related to the spin-3/2 axial anomaly [14].

5. The Index Theorem and Physics

The primary application of the Atiyah-Singer index theorem to physics is in the study of anomalies [2]. These occur in the transition from classical to quantum theories; a symmetry of a classical theory may not be preserved by quantization, leading to changes in the associated identities. In the most extreme cases anomalies can even prevent the construction of a consistent quantum theory, as for the non-abelian anomaly in chiral gauge theories and the conformal anomaly in string theories. Anomalies can often be related to one form or other of the index theorem, and this, besides providing a method for their calculation, shows their essential topological nature. Another important application of the index theorem and of its generalization the G-index theorem is to the analysis of the spectrum of massless particles in Kaluza-Klein theories [3]. The demonstration that the apparent chiral asymmetry of nature could not be produced in a pure Kaluza-Klein theory contributed largely to the demise of such theories.

The original and simplest anomaly is the axial U(1)-anomaly [15]. By using Fujikawa's method [16] the relationship to the index theorem can be easily seen. Let $\psi(x)$ be a Dirac fermion in a 2l-dimensional Riemannian space time with metric $g_{\mu\nu}(x)$, and in the presence of an abelian gauge field $A_\mu(x)$. Then the classical action is

$$S(A,g,\psi,\overline{\psi}) = \int dx\overline{\psi}(x)i\nabla\psi(x)\ , \quad dx = d^{2l}x\sqrt{g}$$

It is invariant under a global chiral transformation,

$$\psi(x) \to e^{i\alpha\Gamma}\psi(x)$$

$$\overline{\psi}(x) \to \overline{\psi}(x)e^{i\alpha\Gamma}\ ,$$

since $\{\Gamma,\nabla\} = 0$. Under a local transformation, $\alpha = \alpha(x)$,

$$S \to S + \int dx\alpha(x)\nabla_\mu j_5^\mu(x)$$

and thus the axial current,

$$j_5^\mu(x) = \overline{\psi}\gamma^\mu\Gamma\psi$$

is conserved classically -

$$\nabla_\mu j_5^\mu(x) = 0.$$

For the quantum theory, Fujikawa's method is to examine the path integral

$$Z(A,g) = \int[d\psi][d\overline{\psi}]\exp(-S(A,g,\psi,\overline{\psi}))$$

This integral should be invariant under a transformation of the integration variables ψ,ψ, provided the change in the path integral measure is taken into account as well as the change in the integrand,

$$\text{i.e.}\ \frac{\delta}{\delta\alpha(x)}Z(A,g) = 0 \tag{10}$$

When the space-time is compact, $i\nabla$ has a discrete spectrum of eigenvalues and $\psi,\overline{\psi}$ can be expanded in terms of a complete set of orthonormal eigenvalues:

$$\psi(x) = \sum_n a_n \psi_n(x)\ , \ \ \overline{\psi}(x) = \sum_n \psi_n^\dagger(x)\overline{b}_n$$

where

$$i\nabla\!\!\!\!/\,\psi_n(x) = \lambda_n \psi_n(x)\ , \ \ \int dx\psi_n^\dagger(x)\psi_m(x) = \delta_{nm}\ .$$

Then the measure can be defined as

$$[d\psi][d\overline{\psi}] = \prod_n d\overline{b}_n da_n\ .$$

Under an infinitesimal local chiral transformation

$$\psi(x) \to \psi(x) + i\alpha(x)\Gamma\psi(x)$$

$$\overline{\psi}(x) \to \overline{\psi}(x) + i\alpha(x)\overline{\psi}(x)\Gamma\ ,$$

$$a_n \to a_n + \sum_m a_m \int dx\ i\alpha(x)\psi_n^\dagger(x)\psi_m(x)$$

$$\overline{b}_n \to \overline{b}_n + \sum_m \overline{b}_m \int dx\ i\alpha(x)\psi_m^\dagger(x)\psi_n(x)$$

and the Jacobian factor is

$$\det{}^{-1}\left(\frac{\partial a_n'}{\partial a_m}\right)\det{}^{-1}\left(\frac{\partial b_n'}{\partial b_m}\right) = \exp(-2i\int dx\ \alpha(x)\sum_n \psi_n^\dagger(x)\Gamma\psi_n(x))\ ,$$

where a suitable regularization is assumed.

Now, if

$$i\nabla\!\!\!\!/\,\psi_n(x) = \lambda_n\psi_n(x)$$

then

$$i\nabla\!\!\!\!/\,(\Gamma\psi_n(x)) = -\lambda_n\Gamma\psi_n(x)\ ,$$

so that, for $\lambda_n \neq 0$,

$$\int dx\psi_n^\dagger(x)\Gamma\psi_n(x) = 0\ .$$

On the other hand, on $\ker(i\nabla\!\!\!\!/\,)$

$$[i\nabla\!\!\!\!/\,,\Gamma] = 0\ ,$$

so that the zero modes of $i\nabla\!\!\!\!/$ can be split into those of positive and those of negative chirality, and thus

$$\int\sum_n \psi_n^\dagger(x)\Gamma\psi_n(x) = \mathrm{ind}(i\nabla_+)\ ,$$

the index of the Dirac operator on the space of positive chirality spinors.

The index theorem (5) can now be used to give the index as a space-time integral, and then for nearly constant α

$$\int\alpha(x)\sum_n\psi_n^\dagger(x)\Gamma\psi_n(x) = \int dx\ \alpha(x)\hat{A}(x)ch(F)$$
$$= \int dx\ \alpha(x)\det{}^{\frac{1}{2}}\left(\frac{\frac{iR}{4\pi}}{\sinh\frac{iR}{4\pi}}\right)\mathrm{tr}\left(\exp\left(\frac{iF}{2\pi}\right)\right).$$

Thus (10) implies that

$$\frac{\delta}{\delta\alpha(x)}\Bigg|_{\alpha=0}\int[d\psi][d\overline{\psi}]\exp(-\int dx\ \alpha(x)(2i\hat{A}(R)ch(F) + \nabla_\mu j_5^\mu(x)))e^{-S} = 0$$

i.e.

$$\left\langle \nabla_\mu j_5^\mu(x)\right\rangle = -2i\hat{A}(R)ch(F)\ . \qquad (11)$$

This is the covariant form of the anomaly.

Physically the correct form for the anomaly is not covariant but rather it must satisfy the Wess-Zumino consistency conditions [17]. An index theorem will always give a covariant expression for the anomaly; however, the consistent anomaly can be obtained from it. In any case, for most applications it is most important that the various anomalies cancel and it is as well to demonstrate this for one form of the anomaly as the other.

Other anomalies which can be derived from an index theorem include the (gauged) non-abelian anomaly for chiral fermions, which comes from the index theorem in (2l+2)-dimensions or equivalently from the family's index theorem [2],[18], and the spin 3/2 anomalies [14] for which the Rarita-Schwinger operator replaces the Dirac operator.

The other application of the Atiyah-Singer index theorem mentioned at the beginning of this section comes about in the following way [3]. Let us begin with a (4+N)-dimensional manifold M_4xB, where M_4 is conventional space-time and B is compact and of the scale of the Planck length, and on which there are defined a metric tensor and matter fields but no elementary Yang-Mills gauge fields. Continuous symmetries of B will manifest themselves in four dimensions as gauge symmetries with the corresponding gauge fields emerging from the metric tensor. The observed matter fields in four dimensions must originate from massless fields on M_4xB as all other fields have masses of the order of the Planck mass. Further, for Dirac fermions the Dirac operator on B will act as a mass operator in the four-dimensional Dirac equation. Thus the spectrum of observed particles must emerge from the zero modes of the Dirac operator on B. Now the observed spectrum is chirally asymmetric both in the relative number of particles of each chirality and inthe representations of the gauge groups which they carry. Thus the G-index of the Dirac operator on B must be non-vanishing. Unfortunately by a theorem of Atiyah and Hirzebruch [19] this is never so. As a result the only physically interesting theories are those such as the superstring theory which begin with elementary gauge fields and matter fields in chirally asymmetric representations.

II. Derivation of the Index Theorem via a Supersymmetric Quantum Mechanics

1. Preamble

There are a number of derivations of the Atiyah-Singer index theorem for the Dirac operator which use the methods of physics. One obvious approach is to turn the relation to the chiral anomaly around and then to calculate the chiral anomaly by some other means. This requires calculations in a quantum field theory on the manifold involved. Simpler methods are based on an analogy between the index of a differential operator between the sections of two vector bundles and the index of a supersymmetry operator between the bosonic and fermionic Hilbert spaces of a supersymmetric quantum mechanics - the N=1/2 supersymmetric non-linear sigma model being the model appropriate for the Dirac operator. The latter index can be written as a trace of the evolution operator and then evaluated. The advantage here is that the calculations are only of a quantum mechanical model, that is a one-dimensional quantum field theory. The usual method of evaluation is via a path integral formulation of the trace [5]-[7], although Zumino [8], referring to the trickiness of the definition of the path integral, opted instead for a WKB calculation. There are also rigorous mathematical evaluations of the trace [22] including one which uses Wiener integrals, the mathematical equivalent of path integrals. As stated at the beginning of this chapter, we will consider the path integral approach. First, though, we review Witten's supersymmetry index itself.

2. Witten's Supersymmetry Index

The concept of the index of an operator was introduced into the study of supersymmetric theories by Witten [4]. He argued that the index of a generator of a supersymmetry must be zero if that supersymmetry is to be broken. Since the spectrum of observed particles does not consist of degenerate boson-fermion pairs, any physically realistic supersymmetric theory must have its supersymmetry broken. This requirement is equivalent to saying that the energy of the vacuum must be non-zero. However, this may be very difficult to check precisely when the symmetry breaking arises from quantum corrections. The requirement that the index be zero, although not a sufficient condition, has the advantage that the index, being a topological quantity, can be calculated exactly in some convenient limit.

To show how the condition comes about it is sufficient to consider a one-dimensional or quantum mechanical model with one supersymmetry generator S acting on the Hilbert space $H_B \oplus H_F$, with

$$S = \begin{pmatrix} 0 & Q^\dagger \\ Q & 0 \end{pmatrix},$$

$$S^2 = H,$$

the Hamiltonian, and

$$(-1)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where F is the fermion number operator. The states of interest, the zero-energy states, arise for a four-dimensional model within the zero- momentum subspace, on which the supersymmetry algebra

$$\{Q^A_\alpha, \overline{Q}_{\dot\beta B}\} = 2\sigma^\mu_{\alpha\dot\beta}\delta^A_B P_\mu$$

reduces to

$$\{Q^A_\alpha, \overline{Q}_{\dot\beta B}\} = 2\delta^A_B \delta_{\alpha\dot\beta} H ,$$

justifying the use of the simpler model.

A symmetry is unbroken if and only if its generator annihilates the vacuum,

$$S\,|0\rangle = 0 .$$

In this case, that implies that the vacuum has zero energy

$$H\,|0\rangle = 0 .$$

On the other hand, if the supersymmetry is broken, then

$$S\,|0\rangle \neq 0$$

and the positivity of $H = S^2 = S^\dagger S$ implies that the vacuum has energy greater than zero. Thus, if there is a state with energy zero, then the vacuum being a state of lowest energy also has energy zero. From the above then, the supersymmetry is broken if and only if

$$\ker H = \ker S = \{0\}$$

and, since

$$\ker S = \ker Q \oplus \ker Q^\dagger ,$$

it is a necessary condition for broken supersymmetry that the index of Q,

$$\mathrm{ind}\, Q = \dim(\ker Q) - \dim(\ker Q^\dagger)$$

be zero. Or, in other words, if the index of Q is not zero, then the supersymmetry must be unbroken and the model is not a physically realistic one.

The index of Q can also be written as the trace over the zero energy states of $(-1)^F$,

$$\mathop{\mathrm{tr}}_{H=0} (-1)^F .$$

A more convenient form would be as a trace over all states, this would be an infinite sum so that care would have to be taken to ensure that it converged. To realize it note that for each non-zero mode $|\psi\rangle$, with

$$H|\psi\rangle = E|\psi\rangle \neq 0 ,$$

there is another state $S|\psi\rangle \neq 0$ with the same energy,

$$HS|\psi\rangle = SH|\psi\rangle = ES|\psi\rangle ,$$

and opposite statistics,

$$(-1)^F S|\psi\rangle = -S(-1)^F|\psi\rangle ,$$

so that

$$\mathrm{ind}\, Q = \mathop{\mathrm{tr}}_{H=0} (-1)^F = \mathrm{tr}((-1)^F e^{-\beta H}) ,\ \ \beta > 0 \tag{12}$$

Provided that the spectrum of H is suitably distributed this infinite sum will be well-defined and independent of $\beta>0$. It provides the basis for the calculations of the index.

3. Formulation of the Path Integral

For a quantum mechanical system with euclidean time and hamiltonian H, the operator $\exp(-\beta H)$ gives the evolution of that system over time β. The path integral representation for matrix elements of the evolution operator in such a system is well-known for conventional time [20], whether the quantum mechanics is given in terms of position and momentum or in terms of bosonic or fermionic creation and annihilation operators. Below we briefly review the euclidean time formulation in its simplest case - a fermionic system with one degree of freedom - arriving at the expression for $\mathrm{tr}((-1)^F\exp(-\beta H))$ as well as for $\mathrm{tr}(\exp(-\beta H))$; we then state the result for the other cases.

A fermionic system with one degree of freedom has a two-dimensional space of states. At a given point in time it has an orthonormal basis $\{|0\rangle, |1\rangle\}$ given in terms of the creation and annihilation operators, a and $\bar{a}$, which satisfy

$$\bar{a} = a^\dagger,\ \{\bar{a},a\} = 1,\ \bar{a}^2 = 0 = a^2,$$

by

$$a|0\rangle = 0,\ \bar{a}|0\rangle = |1\rangle,\ \langle 0|0\rangle = 1.$$

An arbitrary state $|f\rangle$ can be given as either

$$|f\rangle = \alpha|0\rangle + \beta|1\rangle \quad \text{or} \quad |f\rangle = (\alpha + \beta\bar{a})|0\rangle$$

and an operator A as

$$A = \sum_{n,m=0}^{1} |n\rangle A_{nm}\langle m| \quad \text{or} \quad A = \sum_{n,m} A^N_{nm}\bar{a}^n a^m.$$

The analytic functions of a single complex Grassmannian variable, η, provide a realization of this system. The state $|f\rangle$ is given as

$$f(\bar{\eta}) = \alpha + \beta\bar{\eta}$$

and the actions of a and a are realized as

$$\bar{a}f(\bar{\eta}) = \bar{\eta}f(\bar{\eta}) \quad \text{and} \quad af(\bar{\eta}) = \frac{d}{d\bar{\eta}}f(\bar{\eta})\,.$$

The scalar product is

$$\langle f_1|f_2\rangle = \int d\bar{a}da\, e^{-\bar{a}a}\overline{f_1(\bar{a})}f_2(\bar{a})\,.$$

To an operator A can be associated two functions of $\eta,\bar{\eta}$: the kernel

$$A(\bar{\eta},\eta) = \sum_{n,m} A_{nm}\bar{\eta}^n\eta^m\,;$$

and the normal kernel

$$A^N(\bar{\eta},\eta) = \sum_{n,m} A^N_{nm}\bar{\eta}^n\eta^m\,.$$

They are related by

$$A(\bar{\eta},\eta) = e^{\bar{\eta}\eta}A^N(\bar{\eta},\eta)\,.$$

The action of A is given by

$$(Af)(\bar{\eta}) = \int d\bar{\xi}d\eta\; e^{-\bar{\xi}\eta}A(\bar{\eta},\eta)f(\bar{\xi})\,,$$

the trace of A by

$$\mathrm{tr}(A) = \int d\eta d\bar{\eta}\; e^{\bar{\eta}\eta}A(\bar{\eta},\eta) \tag{13}$$

and the product of two operators A_1 and A_2 by

$$(A_1A_2)(\bar{\eta},\eta) = \int d\bar{\xi}d\xi\; e^{-\bar{\xi}\xi}A_1(\bar{\eta},\xi)A_2(\bar{\xi},\eta)\,. \tag{14}$$

Let the hamiltonian be given in normal ordered form, so that we have its normal kernel $h(\eta,\eta) \equiv H^N(\eta,\eta)$. To find $\mathrm{tr}(\exp(-\beta H))$ we require the kernel of $\exp(-\beta H)$. Now splitting the time interval β into N intervals $\Delta t = \beta/N$, with N very large, we have

$$e^{-\beta H} = (e^{-\Delta t H})^N$$

and

$$e^{-\Delta t H} \approx 1 - \Delta t H\,.$$

So the normal kernel of $\exp(-\Delta t H)$ is approximately

$$1 - \Delta t\, h(\bar{\eta},\eta) \approx e^{-\Delta t\, h(\bar{\eta},\eta)}$$

and the kernel

$$e^{\bar{\eta}\eta - \Delta t\, h(\bar{\eta},\eta)}.$$

Then using (13) and (14)

$$\begin{aligned}
\mathrm{tr}(e^{-\beta H}) &= \int d\eta d\bar{\eta}\; e^{\bar{\eta}\eta}(e^{-\beta H})(\bar{\eta},\eta)\\
&= \int d\eta d\bar{\eta}\prod_{i=1}^{N-1}d\bar{\eta}_i d\eta_i\; \exp(\bar{\eta}\eta - \sum_{i=1}^{N-1}\bar{\eta}_i\eta_i + \bar{\eta}\eta_{N-1} - \Delta t\, h(\bar{\eta},\eta_{N-1})\\
&\qquad + \sum_{i=1}^{N-2}(\bar{\eta}_{i+1}\eta_i - \Delta t\, h(\bar{\eta}_{i+1},\eta_i)) + \bar{\eta}_1\eta - \Delta t\, h(\bar{\eta}_1,\eta))\\
&= \int d\eta d\bar{\eta}\prod_{i=1}^{N-1}d\bar{\eta}_i d\eta_i\; \exp(\bar{\eta}(\eta+\eta_{N-1}) - \Delta t\, h(\bar{\eta},\eta_{N-1})\\
&\qquad - \sum_{i=1}^{N-2}(\bar{\eta}_{i+1}(\eta_{i+1}-\eta_i) + \Delta t\, h(\bar{\eta}_{i+1},\eta_i)) - \bar{\eta}_1(\eta_1-\eta) - \Delta t\, h(\bar{\eta}_1,\eta)))
\end{aligned} \tag{15}$$

So that taking $\eta = \eta_N$, $\eta = \eta_0 = -\eta_N$ and the limit $N\to\infty$, $\Delta t\to 0$, we have the path integral representation

$$\mathrm{tr}(e^{-\beta H}) = \int_{APBC}\prod_t d\bar{\eta}(t)d\eta(t)\; \exp(-\int_0^\beta dt\,(\bar{\eta}\dot{\eta} + h(\bar{\eta},\eta))) \tag{16}$$

where APBC stands for anti-periodic boundary conditions. It is to this limit that we will return later to explain the ambiguities arising in the path integral calculations.

We also require $\mathrm{tr}((-1)^F\exp(-\beta H))$. Now

$$(-1)^F = 1 - 2\bar{a}a$$

so that its normal kernel is

$$1 - 2\bar{\eta}\eta_{N-1} = e^{-2\bar{\eta}\eta_{N1}}$$

and its kernel

$$e^{-\bar{\eta}\eta_{N1}} .$$

Thus

$$\mathrm{tr}((-1)^F e^{-\beta H}) = \int d\eta d\bar{\eta} \prod_{i=1}^{N-1} d\bar{\eta}_i d\eta_i \exp(\bar{\eta}(\eta-\eta_{N-1}) - \sum_{i=1}^{N-2}(\bar{\eta}_{i+1}(\eta_{i+1}-\eta_i) + \Delta t\, h(\bar{\eta}_{i+1},\eta_i)) - \bar{\eta}_1(\eta_1-\eta) - \Delta t\, h(\bar{\eta}_1,\eta))$$

so that taking $\bar{\eta} = -\bar{\eta}_N$, $\eta = \eta_N = \eta_0$, we have

$$\mathrm{tr}((-1)^F e^{-\beta H}) = \int_{PBC} \prod_t d\bar{\eta}(t) d\eta(t) \exp(-\int_0^\beta dt\, (\bar{\eta}\dot{\eta} + h(\bar{\eta},\eta))) \qquad (17)$$

where PBC stands for periodic boundary conditions.

All of the above of course generalizes straightforwardly to a fermionic system with multiple degrees of freedom.

For the actual evaluation of such path integrals a different realization of the path integral measure is used - we will use one in terms of modes of Fourier expansions of $\eta(t)$, $\bar{\eta}(t)$ on the interval $[0,\beta]$. For this reason it is important to fix the normalization of these path integrals.

For a fermionic system with d degrees of freedom (16) is easily normalized using

$$\mathrm{tr}(e^{-\beta.0}) = \mathrm{tr}(1) = 2^d$$

i.e.

$$\mathrm{tr}(e^{-\beta H}) = 2^d \frac{\mathrm{tr}(e^{-\beta H})}{\mathrm{tr}(e^{-\beta.0})}$$

Note also that

$$\mathrm{tr}(e^{-\beta H}) = \mathrm{tr}((-1)^F(-1)^F e^{-\beta H})$$
$$= \int_{PBC} \prod_t d\bar{\eta}(t) d\eta(t) \exp(-2\bar{\eta}_0\eta_0)\exp(-\int_0^\beta dt\, (\bar{\eta}\dot{\eta} + h(\bar{\eta},\eta))) .$$

Splitting the integral into a conventional integral over constant modes, $\int d\eta d\bar{\eta}$, and a path integral over non-constant modes which vanish at the boundaries, denoted $\int_0 D\bar{\eta}D\eta$, we have

$$\mathrm{tr}(e^{-\beta H}) = \int d\bar{\eta} d\eta \exp(-2\bar{\eta}\eta) \int_0 D\bar{\eta}D\eta \exp(-\int_0^\beta dt\, (\bar{\eta}\dot{\eta} + h(\bar{\eta},\eta)))$$

and, when $H = 0$,

$$2^d = 2^d \int_0 D\bar{\eta}D\eta \exp(-\int_0^\beta dt\, \bar{\eta}\dot{\eta}) .$$

On the other hand,

$$\mathrm{tr}((-1)^F e^{-\beta H}) = \int d\bar{\eta} d\eta \int_0 D\bar{\eta}D\eta \exp(-\int_0^\beta dt\, (\bar{\eta}\dot{\eta} + h(\bar{\eta},\eta))) .$$

Combining the two,

$$\mathrm{tr}((-1)^F e^{-\beta H}) = \frac{\int d\bar{\eta} d\eta \int_0 D\bar{\eta}D\eta \exp(-\int_0^\beta dt\, (\bar{\eta}\dot{\eta} + h(\bar{\eta},\eta)))}{\int_0 D\bar{\eta}D\eta \exp(-\int_0^\beta \bar{\eta}\dot{\eta})} \qquad (18)$$

The path integrals for bosonic systems are analagous - that is the exponent is in each case $-S_E$, where S_E is the euclidean action corresponding to the given hamiltonian. For a bosonic system defined interms of creation and annihilation operators

$$\mathrm{tr}(e^{-\beta H}) = \int_{PBC} \prod_t \frac{d\bar{\eta}(t)d\eta(t)}{2\pi i} \exp(-\int_0^\beta dt\, (\bar{\eta}\dot{\eta} + h(\bar{\eta},\eta))) \, . \tag{19}$$

In terms of position and momentum

$$\mathrm{tr}(e^{-\beta H}) = \int \prod_t \frac{dp(t)dx(t)}{2\pi} \exp(-\int_0^\beta dt\, (-ip\dot{x} + h(x,p)))$$

where the path integral is over those x(t) with periodic boundary conditions.

When the hamiltonian is of the form

$$h(p,x) = \tfrac{1}{2}(p+f(x))^2 + V(x)$$

the transformation

$$p+f(x)-i\dot{x} \rightarrow p$$

enables the p(t)-integration to be performed, yielding

$$\mathrm{tr}(e^{-\beta H}) = \int_{PBC} \prod_t dx(t) \exp(-\int_0^\beta dt\, (\tfrac{1}{2}\dot{x}^2 + i\dot{x}f(x) + V(x))) \tag{20}$$

Normalization here is with respect to the free hamiltonian $H={}^1\!/_2P^2$. Using

$$\begin{aligned}\mathrm{tr}(e^{-\frac{1}{2}\beta P^2}) &= \int d^dx \int d^dp\, \langle x| e^{-\frac{1}{2}\beta P^2} |p\rangle\langle p|x\rangle \\ &= \int d^dx \int d^dp\, \frac{e^{-\frac{1}{2}\beta p^2}}{(2\pi)^d} \\ &= (2\pi\beta)^{-\frac{d}{2}} \int d^dx \ ,\end{aligned}$$

we have

$$\mathrm{tr}(e^{-\beta H}) = (2\pi\beta)^{-\frac{d}{2}} \frac{\int d^dx \int_0 Dx \exp(-\int_0^\beta dt\, (\frac{1}{2}\dot{x}^2 + i\dot{x}f(x) + V(x)))}{\int_0 Dx \exp(-\int_0^\beta dt\, \frac{1}{2}\dot{x}^2)} \tag{21}$$

4. The Index of the Dirac Operator

Let M be an ($n=2\ell$)-dimensional riemannian manifold on which is defined a spin structure. The index of the Dirac operator over M can be expressed as a path integral in a supersymmetric quantum mechanical system through the identifications:

$$C^{\infty}(\Delta(M)) \leftrightarrow H$$

$$\Gamma \leftrightarrow (-1)^F$$

$$C^{\infty}(\Delta^+(M))\oplus C^{\infty}(\Delta^-(M)) \leftrightarrow H_B\oplus H_F$$

$$\frac{1}{\sqrt{2}} i\nabla\!\!\!/ \leftrightarrow S$$

$$\frac{1}{2}(i\nabla\!\!\!/)^2 \leftrightarrow H$$

That is, by finding a quantum mechanical space of states isomorphic to the space of spinors on M, the index of the Dirac operator can be seen as the index of the corresponding quantum mechanical operator which generates a supersymmetry. Its index can then be expressed through $\mathrm{tr}((-1)^F\exp(-\beta H))$ as a path integral. The fact that this path integral must be independent of $\beta>0$ then enables it to be evaluated.

The appropriate space, H, is the direct product of a position-momentum space with n degrees of freedom and a fermionic space with l degrees of freedom. The position space represents the points of M and the fermionic space the spinor structure. So that $\psi\in C^{\infty}(\Delta(M))$ is given as

$$|\psi\rangle = \int dx \sqrt{g}\, \psi_{j_1\ldots j_\ell}(x)|x\rangle\otimes \bar{a}_1^{j_1}\ldots\bar{a}_1^{j_\ell}|0\rangle \tag{22}$$

The representation of the 2^ℓ-dimensional space of spinors at each point in terms of l fermionic creation operators, a_j, $j=1,\ldots,\ell$, comes about in the following way [10]. Given the complexified Clifford algebra, $C_{2l}\otimes C$ generated by γ^α, $\alpha=1,\ldots,2\ell$, $\{\gamma^\alpha,\gamma^\alpha\}=2\delta^{\alpha\beta}$, the space of spinors can be realized as a left ideal of $C_{2l}\otimes C$,

$$\Delta = \{\psi\in C_{2l}\otimes C;\ Q_j\psi=-\psi,\ j=1,\ldots,\ell\}\ ,$$

where Q_j is right multiplication by $i\gamma^{2j-1}\gamma^{2j}$, $j=1,\ldots,l$. The Clifford algebra action on the spinors is then just Clifford multiplication from the left.

Now writing

$$a_j = \frac{1}{2}(\gamma^{2j-1}+ i\gamma^{2j})$$
$$\bar{a}_j = \frac{1}{2}(\gamma^{2j-1}- i\gamma^{2j})$$

we have

$$\{a_j,\bar{a}_k\} = \delta_{jk}\ ,\ \{\bar{a}_j,\bar{a}_k\} = 0\ ,\ \{a_j,a_k\} = 0\ .$$

Further the 2^l products

$$\prod_{j=1}^{\ell}\{\bar{a}_j, 1-\bar{a}_j a_j\}\ ,$$

where for each j we take one of the terms in brackets, form a basis for the space of spinors. Thus defining

$$|0\rangle = a_1 \ldots a_l$$

so that

$$a_j|0\rangle = 0 \;,\; j=1,\ldots,\ell \,,$$

the space of spinors is the space of states generated from $|0\rangle$ by the ℓ creation operators $\bar{a}_j$.

In this representation

$$\gamma^{2j-1} = a_j + \bar{a}_j$$

$$\gamma^{2j} = -i(a_j - \bar{a}_j) \quad , j=1,\ldots,\ell \,,$$

and

$$\begin{aligned} \Gamma &= i^l \gamma^1 \ldots \gamma^{2\ell} \\ &= \prod_{j=1}^{l} (1 - 2\bar{a}_j a_j) \\ &= \exp(-2\sum_j \bar{a}_j a_j) \\ &= (-1)^F \,. \end{aligned}$$

The Dirac operator is

$$i\nabla\!\!\!/ = \gamma^\alpha E_\mu^\alpha(x)\,(i\partial_\mu + i\omega_\mu(x))$$

where $E_\mu{}^\alpha(x)$ is the inverse vielbein,

$$\delta^{\alpha\beta} E_\alpha^\mu(x) E_\beta^\nu(x) = g^{\mu\nu}(x)$$

and

$$\omega_\mu(x) = \tfrac{1}{8}\,\omega_{\mu\alpha\beta}(x)[\gamma^\alpha,\gamma^\beta]$$

is the spin connection.

$$\begin{aligned} \tfrac{1}{2}(i\nabla\!\!\!/)^2 &= \tfrac{1}{4}\{\gamma^\alpha,\gamma^\beta\} E_\alpha^\mu(x) E_\beta^\nu(x)\,(i\partial_\mu + i\omega_\mu(x))\,(i\partial_\nu + i\omega_\nu(x)) \\ &\qquad - \tfrac{1}{4}\gamma^\alpha\gamma^\beta E_\alpha^\mu(x) E_\beta^\nu(x)\,[\nabla_\mu,\nabla_\nu] \\ &= \tfrac{1}{2} g^{\mu\nu}(x)\,(i\partial_\mu + i\omega_\mu(x))\,(i\partial_\nu + i\omega_\nu(x)) - \tfrac{1}{16}\gamma^\alpha\gamma^\beta E_\alpha^\mu(x) E_\beta^\nu(x)\gamma^\gamma\gamma^\delta R_{\mu\nu\gamma\delta}(x) \,, \end{aligned}$$

using $[\nabla_\mu,\nabla_\nu] = \tfrac{1}{8} R_{\mu\nu\gamma\delta}[\gamma^\gamma,\gamma^\delta]$,

$$= \tfrac{1}{2} g^{\mu\nu}(x)\,(i\partial_\mu + i\omega_\mu(x))\,(i\partial_\nu + i\omega_\nu(x)) + \tfrac{1}{8} R(x) \,,$$

where $R(x) = E_\alpha^\mu(x) E_\beta^\nu(x) R_{\mu\nu}{}^{\alpha\beta}(x)$ is the Ricci scalar.

In the quantum mechanical representation

$$-i\partial_\mu \rightarrow P_\mu$$

so that

$$\tfrac{1}{2}(i\nabla\!\!\!/)^2 \rightarrow H = \tfrac{1}{2} g^{\mu\nu}(X)\,(P_\mu - \tfrac{1}{4} i\omega_{\mu\alpha\beta}(X)\gamma^\alpha\gamma^\beta)\,(P_\nu - \tfrac{1}{4} i\omega_{\nu\gamma\delta}(X)\gamma^\gamma\gamma^\delta) + \tfrac{1}{8} R(X) \,.$$

In order to substitute this hamiltonian into our expressions for the path integral we must ensure that the terms involving a_j, a_j are in normal ordered form (on the other hand, we will assume that the ordering of P and X is correct, see Faddeev [20]). Then defining $\Omega^{\alpha\beta}$ by

$$\Omega^{2j-1,2j} = 1 = -\Omega^{2j,2j-1}$$

and ψ^{α} by

$$\psi^{2j-1} = \frac{1}{\sqrt{2}}(\eta_j + \bar{\eta}_j)$$
$$\psi^{2j} = -\frac{i}{\sqrt{2}}(\eta_j - \bar{\eta}_j) ,$$

we have

$$\begin{aligned} \mathrm{ind}(i\nabla\!\!\!\!/_{+}) &= \mathrm{tr}((-1)^F e^{-\beta H}) \\ &= \int_{PBC} \prod_t \sqrt{g}\, dx^{\mu}(t)\, d\bar{\eta}^j(t)\, d\eta^j(t) \\ &\quad \exp\left[-\int_0^{\beta} dt\, (\tfrac{1}{2} g_{\mu\nu}(x)\dot{x}^{\mu}\dot{x}^{\nu} + \tfrac{1}{2}\dot{x}^{\mu}\omega_{\mu\alpha\beta}(x)\, (\psi^{\alpha}\psi^{\beta} + \tfrac{i}{2}\Omega^{\alpha\beta}) + \bar{\eta}^j\dot{\eta}^j + \tfrac{1}{8}R(x))\right] \end{aligned}$$

Making the change of variable

$$\bar{\eta}^j, \eta^j \to \psi^{\alpha}$$
$$d\bar{\eta}^j(t)\, d\eta^j(t) = -i\, d\psi^{2j-1}(t)\, d\psi^{2j}(t)$$
$$\bar{\eta}^j\dot{\eta}^j = \tfrac{1}{2}(\psi^{2j-1}\dot{\psi}^{2j-1} + \psi^{2j}\dot{\psi}^{2j} + \frac{d}{dt}(\psi^{2j-1}\psi^{2j})) ,$$

$$\begin{aligned} \mathrm{ind}(i\nabla\!\!\!\!/_{+}) &= \int_{PBC} \prod_t \sqrt{g}\, dx^{\mu}(t)\, (-i)^l d\psi^{\alpha}(t) \\ &\exp\left[\tfrac{1}{2}\int_0^{\beta} dt\, (g_{\mu\nu}(x)\dot{x}^{\mu}\dot{x}^{\nu} + \delta_{\alpha\beta}\psi^{\alpha}\dot{\psi}^{\beta} + \psi^{\alpha}\dot{x}^{\mu}\omega_{\mu\alpha\beta}(x)\psi^{\beta} + \tfrac{i}{2}\dot{x}^{\mu}\omega_{\mu\alpha\beta}(x)\Omega^{\alpha\beta} + \tfrac{1}{4}R(x))\right] \end{aligned} \tag{23}$$

Where the curved background introduces $\sqrt{g}$ into the measure for $x^{\mu}(t)$.

The evaluation of the integral proceeds through the separation of the integration variables $x^{\mu}(t)$, $\psi^{\alpha}(t)$ into constant and non-constant modes as in [5],[6]

$$x^{\mu}(t) = x_0^{\mu} + u^{\mu}(t) ,\ u^{\mu}(0) = 0 = u^{\mu}(\beta)$$
$$\psi^{\alpha}(t) = \psi_0^{\alpha} + \xi^{\alpha}(t) ,\ \xi^{\alpha}(0) = 0 = \xi^{\alpha}(\beta) ,$$

with the normalization of the integral as given earlier. That is

$$\begin{aligned} \mathrm{ind}(i\nabla\!\!\!\!/_{+}) &= (-i)^l \int d\psi_0^{\alpha}\, (2\pi\beta)^{-l} \int dx_0^{\mu} \sqrt{g(x_0)} \int_0 D\xi^{\alpha} \int_0 Du^{\mu} \\ &\exp\left[-\tfrac{1}{2}\int_0^{\beta} dt\, (g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} + \delta_{\alpha\beta}\psi^{\alpha}\dot{\psi}^{\beta} + \psi^{\alpha}\dot{x}^{\mu}\omega_{\mu\alpha\beta}(x)\psi^{\beta} + \tfrac{i}{2}\dot{x}^{\mu}\omega_{\mu\alpha\beta}\Omega^{\alpha\beta} + \tfrac{1}{4}R(x))\right] \\ &\times\left(\int_0 D\xi^{\alpha} \exp\left[-\tfrac{1}{2}\int_0^{\beta} dt\, \delta_{\alpha\beta}\xi^{\alpha}\dot{\xi}^{\beta}\right] \int_0 Du^{\alpha} \exp\left[-\tfrac{1}{2}\int_0^{\beta} dt\, \delta_{\alpha\beta}\dot{u}^{\alpha}\dot{u}^{\beta}\right]\right)^{-1} \end{aligned}$$

where $Du^{\mu} = \prod_t du^{\mu}(t)\sqrt{g(x_0+u(t))}$.

Under the rescaling $t \to \beta t$, $\psi_0 \to \beta^{-\frac{1}{2}}\psi_0$, $u(t) \to \beta^{\frac{1}{2}}u(t)$,

$$d\psi_0^\alpha \to \beta^1 d\psi_0^\alpha$$
$$Du^\mu \to \prod_t \beta^1 du^\mu(t)\,(\sqrt{g(x_0)} + O(\beta^{\frac{1}{2}}))$$
$$Du^\alpha \to \prod_t \beta^1 du^\alpha(t)\ ,$$

the measure becomes

$$(2\pi i)^{-1}\int d\psi_0^\alpha \int dx_0^\mu \sqrt{g(x_0)} \int_0 D\xi^\alpha \int \prod_t du^\mu(t)\sqrt{g(x_0)}$$
$$\times\left(\int_0 D\xi^\alpha \exp\left[-\tfrac{1}{2}\int_0^1 dt\ \delta_{\alpha\beta}\xi^\alpha\dot{\xi}^\beta\right]\int_0 Du^\alpha \exp\left[-\tfrac{1}{2}\int_0^1 dt\ \delta_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta\right]\right)^{-1} + O(\beta^{\frac{1}{2}}) \qquad (24)$$

and the exponent becomes

$$\tfrac{1}{2}\int_0^1 dt\ (g_{\mu\nu}(x_0)\dot{u}^\mu\dot{u}^\nu + \delta_{\alpha\beta}\xi^\alpha\dot{\xi}^\beta + \dot{u}^\mu\dot{u}^\nu\omega_{\mu,\nu\,\alpha\beta}(x_0)\psi_0^\alpha\psi_0^\beta$$
$$+ 2\psi_0^\alpha\dot{u}^\mu\omega_{\mu\alpha\beta}(x_0)\psi^\beta(t)) + O(\beta^{\frac{1}{2}}) \qquad (25)$$

Now recalling that the index must be independent of β, we may ignore the terms of order $\sqrt{\beta}$. Note that then the terms

$$\tfrac{i}{2}\dot{x}^\mu\omega_{\mu\alpha\beta}\Omega^{\alpha\beta} + \tfrac{1}{4}R(x)\ ,$$

which arose from a correct consideration of operator ordering but were neglected by other authors [5],[6], do not contribute as we would expected.

If we also make the substitution [5]

$$\xi^\alpha(t) \to \xi^\alpha(t) - \omega_{\mu\ \beta}^{\ \alpha}(x_0)\psi_0^\beta u^\mu(t)\ ,$$

then the measure (24) is unchanged and (25) becomes

$$-\tfrac{1}{2}\int_0^1 dt\ (g_{\mu\nu}(x_0)\dot{u}^\mu\dot{u}^\nu + \tfrac{1}{2}R_{\mu\nu\alpha\beta}(x_0)u^\mu\dot{u}^\nu\psi_0^\alpha\psi_0^\beta + \delta_{\alpha\beta}\xi^\alpha\dot{\xi}^\beta)$$

and, setting

$$u^\alpha(t) = e_\mu^\alpha(x_0)u^\mu(t)\ ,$$

we arrive at the final path integral expression:

$$\mathrm{ind}(i\nabla\!\!\!\!/_+) = (2\pi i)^{-1}\int d\psi_0^\alpha \int dx_0^\mu \sqrt{g(x_0)}\left(\int Du^\alpha \exp\left[\tfrac{1}{2}\int_0^1 dt\ u_\alpha\frac{d^2}{dt^2}u^\alpha\right]\right)^{-1}$$
$$\int_0 Du^\alpha \exp\left[\tfrac{1}{2}\int_0^1 dt\ u^\alpha\left(\frac{d^2}{dt^2}\delta_{\alpha\beta} - \tfrac{1}{2}R_{\alpha\beta\gamma\delta}(x_0)\psi_0^\gamma\psi_0^\delta\frac{d}{dt}\right)u^\beta\right] \qquad (26)$$

That is the only path integral remaining to be evaluated is "gaussian".

The ordinary gaussian integral is

$$\int dx\ e^{-x^tAx} \propto \det^{-\frac{1}{2}}A$$

In (26) the matrix of which we must take the determinant is infinite dimensional and care must be taken. Expanding $u^\alpha(t)$ in a Fourier series

$$u^\alpha(t) = \sum_{n=1}^{\infty}(a^\alpha\sqrt{2}\cos(2n\pi t) + b^\alpha\sqrt{2}\sin(2n\pi t))$$

and writing formally

$$\tfrac{1}{2}R_{\alpha\beta\gamma\delta}\psi_0^\gamma\psi_0^\delta = \begin{pmatrix} 0 & x_1 & & \\ -x_1 & 0 & & \\ & & \ddots & \\ & & 0 & x_1 \\ & & -x_1 & 0 \end{pmatrix},$$

$\frac{d^2}{dt^2}\delta_{\alpha\beta} - \frac{1}{2}R_{\alpha\beta\gamma\delta}\psi_0^\gamma\psi_0^\delta$ takes the nearly diagonal form

$$\bigoplus_{n=1}^{\infty}\bigoplus_{i=1}^{1}\begin{pmatrix} 0 & 0 & -2\pi n & -x_i \\ 0 & 0 & x_i & -2\pi n \\ 2\pi n & -x_i & 0 & 0 \\ x_i & 2\pi n & 0 & 0 \end{pmatrix}$$

and

$$\det^{-\frac{1}{2}}\left(\frac{\frac{d}{dt^2}\delta_{\alpha\beta} - \frac{1}{2}R_{\alpha\beta\gamma\delta}\psi_0^\gamma\psi_0^\delta}{\frac{d}{dt^2}\delta_{\alpha\beta}}\right) = \left(\prod_{n=1}^{\infty}\prod_{i=1}^{1}\left(1 + \left(\frac{x_i}{2\pi n}\right)^2\right)^2\right)^{-\frac{1}{2}}$$

$$= \prod_{i=1}^{1}\prod_{n=1}^{\infty}\left(1 + \left(\frac{x_i}{2\pi n}\right)^2\right)^{-1}$$

This infinite product is standard and its limit is

$$= \prod_{i=1}^{1}\left(\frac{\sinh\frac{x_i}{2}}{\frac{x_i}{2}}\right)^{-1}$$

$$= \det^{-\frac{1}{2}}\left(\frac{\sinh\left(\frac{1}{4}R_{\alpha\beta\gamma\delta}\psi_0^\gamma\psi_0^\delta\right)}{\frac{1}{4}R_{\alpha\beta\gamma\delta}\psi_0^\gamma\psi_0^\delta}\right)$$

Thus

$$\operatorname{ind}(i\nabla\!\!\!\!/_+) = (2\pi i)^{-1}\int d\psi^\alpha\int dx^\mu\sqrt{g(x)}\,\det^{-\frac{1}{2}}\left(\frac{\sinh\left(\frac{1}{4}R_{\alpha\beta\gamma\delta}\psi^\gamma\psi^\delta\right)}{\frac{1}{4}R_{\alpha\beta\gamma\delta}\psi^\gamma\psi^\delta}\right) \qquad (27)$$

Changing $\psi^\alpha \to \left(\frac{i}{2\pi}\right)^{\frac{1}{2}}\psi^\alpha$, noting that $\int d\psi^\alpha\, \psi^{\alpha_1}\ldots\psi^{\alpha_n} = (-1)^1\varepsilon^{\alpha_1\ldots\alpha_n}$

and that $\varepsilon^{\alpha_1\ldots\alpha_n}\int dx^\mu\sqrt{g}\, f(x) = \int_M f(x)e^{\alpha_1}\wedge\ldots\wedge e^{\alpha_n}$, where $e^\alpha = e^\alpha_\mu dx^\mu$,

and denoting $R_{\alpha\beta\gamma\delta}e^\gamma\wedge e^\delta$ by R, we find that (27) is just

$$\mathrm{ind}(i\nabla\!\!\!/_+) = \int_M \det^{-\frac{1}{2}}\left(\frac{\sinh\frac{iR}{4\pi}}{\frac{iR}{4\pi}}\right) = \int_M \hat{A}(M)\,,$$

which is the Atiyah-Singer index theorem result for the Dirac operator (5).

5. The Index of the Twisted Dirac Operator

We now consider the index of the twisted Dirac operator,

$$i\nabla\!\!\!/_{V+}: C^\infty(\Delta^+(M)\otimes V) \to C^\infty(\Delta^-(M)\otimes V)\,,$$

i.e. the index of the Dirac operator in the presence of gauge as well as gravitational fields. Let G be the group of V, the gauge group, φ the representation involved and d the dimension of φ.

The quantum mechanical space of states isomorphic to $C^\infty(\Delta(M)\otimes V)$ consists of states

$$|\psi\rangle = \int dx\sqrt{g}\,\psi^a_{j_1\ldots j_l}(x)|x\rangle\otimes \bar{a}_1^{j_1}\ldots\bar{a}_l^{j_l}|0\rangle\otimes\bar{c}_a|0\rangle$$
$$\equiv |\psi^a\rangle\otimes\bar{c}_a|0\rangle$$

where ψ^a carries the representation φ, and each $|\psi^a\rangle$, a=1,...,d, is an element of $C^\infty(\Delta(M))$ as given in the last section. The $\bar{c}_a$ generate a fermionic (although bosonic would do just as well) space of states through

$$\{\bar{c}_a,\bar{c}_b\} = 0,\ \{c^a,c^b\} = 0,\ \{\bar{c}_a,c^b\} = \delta_a^b,\ c^a|0\rangle\,.$$

Note that in order to consider only spinors ψ^a carrying the representation φ, and not all of its anti-symmetric (symmetric) products as well, the space of states must be restricted to the "one-particle" set of states - those generated from the vacuum by just one c_a.

The twisted Dirac operator is

$$i\nabla\!\!\!/_V = i\gamma^\alpha E^\mu_\alpha(x)\,(\partial_\mu + \omega_\mu(x) + A_\mu(x))$$

where ${A_\mu}^a{}_b(x)$ is the Yang-Mills connection taking values in the representation of the Lie algebra, *g*, of G corresponding to φ. From now on we will supress the gravitational terms as they do not mix with the Yang-Mills terms.

$$\tfrac{1}{2}(i\nabla\!\!\!/_V)^2 = -\tfrac{1}{2}\delta^{\mu\nu}\,(\partial_\mu + A_\mu(x))\,(\partial_\nu + A_\nu(x)) - \tfrac{1}{4}\gamma^\mu\gamma^\nu[\nabla_\mu,\nabla_\nu]$$
$$= \tfrac{1}{2}\delta^{\mu\nu}\,(i\partial_\mu + iA_\mu(x))\,(i\partial_\nu + iA_\nu(x)) - \tfrac{1}{4}\gamma^\mu\gamma^\nu F_{\mu\nu}(x)$$

where $F_{\mu\nu}(x)$ is the Yang-Mills curvature corresponding to $A_\mu(x)$.

In the quantum mechanical representation, an element of $\varphi(g)$, $T^a{}_b$ is replaced by $\bar{c}_a T^a{}_b c^b$ so that

$$\bar{c}_a T^a{}_b c^b \psi^c \bar{c}_c |0\rangle = T^a{}_b \psi^b \bar{c}_a |0\rangle .$$

Representing x^μ, $-i\partial_\mu$, γ^α as before

$$\tfrac{1}{2}(i\nabla\!\!\!\!/_V)^2 \to H = \tfrac{1}{2}(P_\mu - i\bar{\eta}_a A_\mu{}^a{}_b(X)\eta^b)^2 - \tfrac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu}(X) .$$

The index is again given by a trace,

$$\mathrm{ind}(i\nabla\!\!\!\!/_{V+}) = \mathrm{tr}_1((-1)^F e^{-\beta H}) , \tag{28}$$

where the subscript 1 reminds us of the restriction to the one particle states of the c-fermions, and $\Gamma \to (-1)^F$, that is F is here the number operator only for the a-fermions generating the spinor space.

There is now a problem in constructing a path integral formulation for this trace, as the standard procedure requires that the trace be over a full set of states, not just the one-particle states. Some authors [5] have found it sufficient to merely attach an apostrophe to the relevant integral and proceed with the path integral manipulation untroubled. In a situation such as this, where the correct answer is already known anyway, such an ill-defined approach seems unsatisfactory. An alternative approach [6] is as follows.

Extend the quantum mechanical space of states to the full set, but rather than taking

$$\mathrm{tr}((-1)^F e^{-\beta H}) \quad \text{take} \quad \mathrm{tr}((-1)^F e^{-\beta H + i\alpha N_c})$$

where $N_c = \sum_c \bar{c}_a c^a$ is the number operator for the c-fermions. Then on the one hand

$$\mathrm{tr}((-1)^F e^{-\beta H + i\alpha N_c}) = \sum_{n=0}^{\mathrm{d}} e^{in\alpha}\,\mathrm{tr}_n((-1)^F e^{-\beta H}) \tag{29}$$

where tr_n is the trace over the n-particle states for the c-fermions, and, on the other hand, the full trace may be written as a path integral with modified hamiltonian,

$$H' = H - \frac{i\alpha}{\beta} N_c$$

The situation is not quite that of the Witten index, however, the arguments for independence of β still apply since $\{(-1)^F, S\} = 0$ and $[S, H'] = 0$ still hold. After the path integral has been evaluated, it may then be expanded as a power series in $\exp(i\alpha)$ and the index $\mathrm{tr}_1((-1)^F \exp(-\beta H))$ extracted. Note also that the n=0 term in (29), $\mathrm{tr}_0((-1)^F \exp(-\beta H))$ is just the index for the ordinary Dirac operator.

Writing $\eta^a, \bar{\eta}_a$ for the Grassmann variables corresponding to $c^a, \bar{c}_a$ in the path integral,

$$\mathrm{tr}((-1)^F e^{-\beta H'}) = \int_{PBC} \prod_t dx^\mu(t)(-i)^1 d\psi^\alpha(t) \int_{APBC} \prod_t d\bar\eta_a(t) d\eta^a(t)$$
$$\exp\left[-\int_0^\beta dt\, (\tfrac{1}{2}\dot x^2 + \dot x^\mu \bar\eta_a A_\mu{}^a{}_b(x)\eta^b - \tfrac{1}{2}\psi^\mu\psi^\nu\bar\eta_a F_{\mu\nu}{}^a{}_b(x)\eta^b - \tfrac{i}{4}\Omega^{\mu\nu}\bar\eta_a F_{\mu\nu}{}^a{}_b(x)\eta^b - \tfrac{i\alpha}{\beta}\bar\eta_a\eta^a + \bar\eta_a\dot\eta^a + \tfrac{1}{2}\psi_\alpha\dot\psi^\alpha)\right] \tag{30}$$

where anti-periodic boundary conditions arise for the $\eta,\bar\eta$ integrations and $\Omega^{\mu\nu}$ is as in the previous section.

Expanding

$$x^\mu(t) = x_0^\mu + u^\mu(t)\,,\quad \psi^\alpha(t) = \psi_0^\alpha + \xi^\alpha(t)\,,$$

rescaling

$$t \to \beta t\,,\quad u^\mu \to \beta^{\frac{1}{2}} u^\mu\,,\quad \psi_0^\alpha \to \beta^{-\frac{1}{2}}\psi_0^\alpha$$

and dropping terms of order $\sqrt\beta$, as in the last section,

$$\mathrm{tr}((-1)^F e^{-\beta H'}) = (2\pi i)^{-1}\left(\int_0 Du^\mu \exp\left[-\tfrac{1}{2}\int_0^1 dt\,\dot u^2\right]\int_0 D\xi^\alpha \exp\left[-\tfrac{1}{2}\int_0^1 dt\,\xi^\alpha\dot\xi_\alpha\right]\right)^{-1}$$
$$\times \int dx_0^\mu\sqrt g \int d\psi_0^\alpha \int_0 Du^\mu \int_0 D\xi^\alpha \int_{APBC}\prod_t d\bar\eta_a(t)d\eta^a(t)$$
$$\exp\left[-\int_0^1 dt\,(\tfrac{1}{2}g_{\mu\nu}(x_0)\dot u^\mu\dot u^\nu + \tfrac{1}{2}R_{\mu\nu\alpha\beta}(x_0)u^\mu\dot u^\nu\psi_0^\alpha\psi_0^\beta + \tfrac{1}{2}\xi_\alpha\dot\xi^\alpha + \bar\eta_a\dot\eta^a - \tfrac{1}{2}\psi_0^\mu\psi_0^\nu\bar\eta_a F_{\mu\nu}{}^a{}_b(x_0)\eta^b - i\alpha\bar\eta_a\eta^a)\right] \tag{31}$$

where the gravitational terms have been reinstated. Taking care of the ξ^α and u^μ integrations as before and denoting $\frac{1}{2}\psi_0^\mu\psi_0^\nu F_{\mu\nu}{}^a{}_b(x_0)$ by $\mathcal{F}^a{}_b$, we have

$$= (2\pi i)^{-1}\int dx_0^\mu\sqrt g\int d\psi_0^\alpha\,\det{}^{-\frac{1}{2}}\left[\frac{\sinh\frac{R}{2}}{\frac{R}{2}}\right]$$
$$\times\int_{APBC}\prod_t d\bar\eta_a(t)d\eta^a(t)\,\exp\left[-\tfrac{1}{2}\int_0^1 dt\,\bar\eta_a\left(\frac{d}{dt}\delta_b^a - \mathcal{F}^a{}_b - i\alpha\right)\eta^b\right] \tag{32}$$

Once again the independence of β given by the supersymmetry allows the original path integral to be reduced to a product of gaussian path integrals. Formally diagonalize the anti-hermitian matrix

$$\mathcal{F}^a{}_b = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_d\end{pmatrix}$$

and expand $\eta^a(t), \bar\eta_a(t)$ in a complex, anti-periodic Fourier series so as to diagonalize $\frac{d}{dt}$,

$$\eta^a(t) = \sum_{n=-\infty}^{\infty} \eta_n^a \, e^{(2n+1)i\pi t}$$
$$\bar{\eta}_a(t) = \sum_{n=-\infty}^{\infty} \bar{\eta}_{na} \, e^{-(2n+1)i\pi t}$$

The η,η integral then becomes

$$\int \prod_{n,a} d\bar{\eta}_{na} d\eta_n^a \exp\left[-\sum_{\substack{n=-\infty \\ a=1}}^{\substack{\infty \\ d}} ((2n+1)i\pi - z_a - i\alpha)\, \bar{\eta}_{na}\eta_n^a \right] = \prod_{n,a} ((2n+1)i\pi - z_a - i\alpha)$$

Unfortunately, even after normalizing with respect to the free integral, this product is divergent.

$$\frac{\det_{APBC}\left(\frac{d}{dt} - F_{ab} - i\alpha\right)}{\det_{APBC}\left(\frac{d}{dt}\right)} = \prod_{n,a}\left(1 + i\frac{z_a + i\alpha}{(2n+1)\pi}\right) \tag{33}$$

has the same convergence properties as $\sum_{n=-\infty}^{\infty} \frac{1}{2n+1}$.

Using some regularization procedure we could re-arrange the terms of the product so that we had

$$\prod_{a=1}^{d}\prod_{n=0}^{\infty}\left(1 + i\frac{z_a+i\alpha}{(2n+1)\pi}\right)\left(1 + i\frac{z_a+i\alpha}{(2(-n-1)+1)\pi}\right) = \prod_{a=1}^{d}\prod_{n=0}^{\infty}\left(1 + \frac{(z_a+i\alpha)^2}{(2n+1)^2\pi^2}\right).$$

This product is well-defined and

$$= \prod_{a=1}^{d} \cosh\left(\frac{z_a+i\alpha}{2}\right)$$
$$= \det\left(\cosh\left(\tfrac{1}{2}(F + i\alpha)\right)\right).$$

However, <u>this does not give the correct form for the index</u>, indeed it may not even be expanded in integer powers of $\exp(i\alpha)$.

Other re-arrangements of the terms of the product will yield other results. Let us examine the path integral itself in order to find the origins and the extent of this ambiguity.

Let us consider a fermionic system with one degree of freedom and hamiltonian $H = -z\bar{a}a$. The trace of exp(-H) is easily evaluated in this case. The space of states is spanned by $|0\rangle$ and $\bar{a}|0\rangle$, and, using

$$(\bar{a}a)^n = \bar{a}a\,,$$
$$\exp(z\bar{a}a) = 1 + (e^z - 1)\bar{a}a$$

and

$$\mathrm{tr}(\exp(z\bar{a}a)) = 2 + (e^z - 1)$$
$$= 1 + e^z\,. \tag{34}$$

On the other hand, section II.2 enables us to write

$$\mathrm{tr}(\exp(z\bar{a}a)) = \int_{APBC} \prod_t d\bar{\eta}(t) d\eta(t) \exp\left[-\int_0^1 dt\, (\bar{\eta}\dot{\eta} - z\bar{\eta}\eta) \right]$$

Attempting to evaluate this path integral without recourse to its origins leads as above to

$$\prod_{n=-\infty}^{\infty}\left(1+\frac{iz}{(2n+1)\pi}\right) \qquad (35)$$

which is ambiguous.

We may, however, make the following arguments to restict its value [6]. We assume that it is an entire function of z, that is analytic over the whole complex plane, and denote it by f(z). Now, the product is only zero if one of its terms is zero, that is f(z) has zeros only at $z = i(2n+1)\pi$. This immediately implies that

$$f(z) = e^{g(z)}\cosh\left(\frac{z}{2}\right) \qquad (36)$$

where g(z) is an entire function, determined by the behaviour of f(z) as $|z|\to\infty$.

From (35),

$$\frac{f'(z)}{f(z)} = \sum_{n=-\infty}^{\infty}\frac{1}{z-i(2n+1)\pi}\ ,$$

assuming that such manipulations are meaningful, as well as by (36) being

$$= g'(z) + \frac{1}{2}\tanh\left(\frac{z}{2}\right).$$

Its seems reasonable to assume that this is bounded as $|z|\to\infty$ away from $z=i(2n+1)\pi$, and thus that g'(z) is bounded. Therefore g'(z) , being bounded and entire, is constant: g'(z)=b, say, and

$$g(z)=a+bz,\quad f(z) = Ne^{bz}\cosh\left(\frac{z}{2}\right)$$
$$= \frac{N}{2}e^{(b-\frac{1}{2})z}(1+e^{z})$$

Thus the path integral is determined only up to an overall normalization, and, assuming b to be real and z imaginary, a phase.

The origins of this ambiguity are in the discrete to continuum limit used to derive the path integral. We may confirm explicitly that the discrete integral formulation of the trace is exact. That is (15)

$$\lim_{N\to\infty}\int d\eta d\bar{\eta}\prod_{i=1}^{N-1}d\bar{\eta}_i d\eta_i \exp\left[\bar{\eta}(\eta+\eta_{N-1}) + \frac{z}{N}\bar{\eta}\eta_{N-1} - \sum_{i=1}^{N-2}\left(\bar{\eta}_{i+1}(\eta_{i+1}-\eta_i) - \frac{z}{N}\bar{\eta}_{i+1}\eta_i\right)\right.$$
$$\left. - \bar{\eta}_1(\eta_1-\eta) + \frac{z}{N}\bar{\eta}_1\eta\right]$$

$$= \lim_{N\to\infty}\int(-1)^{N-1}d\eta d\bar{\eta}\prod_{i=1}^{N-1}d\eta_i d\bar{\eta}_i \exp\left[\bar{\eta}\eta + \bar{\eta}\eta_{N-1}(1+\tfrac{z}{N}) - \bar{\eta}_1\eta_1 + \bar{\eta}_1\eta(1+\tfrac{z}{N})\right.$$
$$\left. + \sum_{i=1}^{N-2}\left(\bar{\eta}_{i+1}\eta_i(1+\tfrac{z}{N}) - \bar{\eta}_{i+1}\bar{\eta}_{i+1}\right)\right]$$

$$= \lim_{N\to\infty} (-1)^{N-1} \det \begin{pmatrix} 1 & & & 1+\frac{z}{N} \\ 1+\frac{z}{N} & -1 & & \\ & \ddots & \ddots & \\ & & 1+\frac{z}{N} & -1 \end{pmatrix}$$

$$= \lim_{N\to\infty} \left(1 + (1+\tfrac{z}{N})^N\right)$$

$$= 1 + e^z$$

$$= \mathrm{tr}(\exp(z\bar{a}a))$$

On the other hand, other discrete integrals seemingly also converge to

$$\int_{APBC} \prod_t d\bar{\eta}(t) d\eta(t) \exp\left[-\int_0^1 dt\, (\bar{\eta}\dot{\eta} - z\bar{\eta}\eta)\right]$$

such as

$$\lim_{N\to\infty} \int d\eta d\bar{\eta} \prod_{i=1}^{N-1} d\bar{\eta}_i d\eta_i \exp\Big[\bar{\eta}(\eta+\eta_{N-1}) + \tfrac{z}{2N}\bar{\eta}(\eta+\eta_{N-1})$$
$$- \sum_{i=1}^{N-2}\left(\bar{\eta}_{i+1}(\eta_{i+1}-\eta_i) - \tfrac{z}{2N}\bar{\eta}_{i+1}(\eta_{i+1}+\eta_i)\right) - \bar{\eta}_1(\eta_1-\eta) + \tfrac{z}{2N}\bar{\eta}_1(\eta_1+\eta)\Big]$$

$$= \lim_{N\to\infty} (-1)^{N-1} \det \begin{pmatrix} 1+\frac{z}{2N} & & & 1+\frac{z}{2N} \\ 1+\frac{z}{2N} & -1+\frac{z}{2N} & & \\ & \ddots & \ddots & \\ & & 1+\frac{z}{2N} & -1+\frac{z}{2N} \end{pmatrix}$$

$$= \lim_{N\to\infty} \left((1+\tfrac{z}{2N})(1-\tfrac{z}{2N})^{N-1} + (1+\tfrac{z}{2N})^N\right)$$

$$= e^{-\frac{z}{2}} + e^{\frac{z}{2}}$$

$$= \cosh(\tfrac{z}{2})$$

Thus it seems that <u>information is lost in the change to the path integral</u> and, taking these two examples as a guide, the ambiguity is as derived earlier.

Returning to the derivation of the index, we have

$$\int_{APBC} \prod_t d\bar{\eta}_a(t) d\eta^a(t) \exp\left[-\int_0^1 dt\, \bar{\eta}_a\left(\frac{d}{dt}\delta^a_b - F^a{}_b - i\alpha\delta^a_b\right)\eta^b\right]$$

$$= \det\left(\frac{N}{2} e^{(b-\frac{1}{2})(F+i\alpha)}(1 + e^{F+i\alpha})\right)$$

In order to determine what values N and b should take in this particular case, Windey [6] assumes that this path integral should take the same value as the corresponding trace, that is

$$\mathrm{tr}\left(\exp(\bar{c}_a(F^a{}_b + i\alpha)c^b)\right)$$

However, if, as we have seen above, the ambiguity arises from the many to one nature of the change from the discrete to the path integral, then after manipulations have been performed upon the path integral we are not justified in returning through a discrete integral to a trace - that is unless we write

$$\det\left(\frac{N}{2}\exp((b-\tfrac{1}{2})(F+i\alpha))\right)\mathrm{tr}\left(\exp(\bar{c}_a(F^a{}_b+i\alpha\delta^a{}_b)c^b)\right)$$

Singh and Steiner [21] give a simple example of how a manipulation of a path integral may lead to incorrect results. Using integration by parts

$$\int_0^1 dt\,\bar{\eta}\dot{\eta} = -\int_0^1 dt\,\dot{\bar{\eta}}\eta$$

but, on the other hand,

$$\sum_i \bar{\eta}_{i+1}(\eta_{i+1}-\eta_i) \neq -\sum_i (\bar{\eta}_{i+1}-\bar{\eta}_i)\eta_{i+1}\,.$$

They go on to evaluate the matrix elements of the time evolution operator for the quite simple hamiltonian which they are considering using only the discrete formulation. Unfortunately, such an approach is not practical in the case with which we are concerned due to the complexity of our hamiltonian. Any attempt [9] to evaluate the index through a discrete formulation must become hopelessly bogged down in the algebra. The reduction of the initial path integral into integrals over constant modes and gaussian integrals over non-constant modes does not have an obvious equivalent in the discrete formulation.

In fact in this case we can fix the values that N and b should take in order that

$$\mathrm{tr}((-1)^F e^{-\beta H'}) = (2\pi i)^{-1}\int dx^\mu\sqrt{g}\int d\psi^\alpha \det{}^{-\frac{1}{2}}\left[\frac{\sinh\frac{R}{2}}{\frac{R}{2}}\right]$$
$$\det\left[\frac{N}{2}\exp((b-\tfrac{1}{2})(F+i\alpha))\left(1+\exp(F+i\alpha)\right)\right] \quad (37)$$

be true, without appealing to any a posteriori argument. We saw at the beginning (29), that $\mathrm{tr}((-1)^F e^{-\beta H'})$ also equal to

$$\sum_{n=0}^{d} e^{in\alpha}\mathrm{tr}_n((-1)^F e^{-\beta H})\,,$$

with the coefficient of $e^{i\alpha}$ being the index which we are seeking, and the constant (n=0) term being the index for the ordinary Dirac operator, which we have already found.

Now (37) may only be expanded as a series in positive integral powers of $e^{i\alpha}$ if $b={}^1/_2, {}^3/_2, \ldots$

$$\det\left[\frac{N}{2}\exp((b-\tfrac{1}{2})(F+i\alpha))\left(1+\exp(F+i\alpha)\right)\right] = \left(\frac{N}{2}\right)^d e^{(b-\frac{1}{2})d\alpha}\det e^{(b-\frac{1}{2})F}\left(1+e^{i\alpha}\mathrm{tr}\,e^F+\ldots\right)$$

and if the n=0 term is not to vanish then of course we must have $b={}^1/_2$. N=2 immediately follows since the remainder of (37) is already $\mathrm{ind}(i\nabla\!\!\!\!/_+)$ correctly normalized.

Thus

$$\mathrm{tr}((-1)^F e^{-\beta H'}) = \int_M \det^{-\frac{1}{2}}\left[\frac{\sinh\frac{iR}{4\pi}}{\frac{iR}{4\pi}}\right]\det\left[1+\exp\left(\frac{iF}{2\pi}+i\alpha\right)\right] \tag{38}$$

where $F = F_{\mu\nu}\, dx^\mu \wedge dx^\nu$, and the index of the twisted Dirac operator is

$$\mathrm{ind}(i\nabla\!\!\!\!/_{V+}) = \int_M \det^{-\frac{1}{2}}\left[\frac{\sinh\frac{iR}{4\pi}}{\frac{iR}{4\pi}}\right]\mathrm{tr}\left[\exp\left(\frac{iF}{2\pi}\right)\right],$$

which is the result of the Atiyah-Singer index theorem (5).

Note that the value that b must take is in the end consistent with the final $\eta,\bar{\eta}$ integration being given as a trace - indeed Windey would not have made the assumption if it had not led to the correct answer! While the reason for this is not clear, it may be that after such manipulations as were performed the trace interpretation will always lead to the desired answer.

Finally, note that the same problems would have arisen for bosonic $c,\bar{c}$ and $\eta,\bar{\eta}$. We would then have $\det^{-1}_{PBC}\left(\frac{d}{dt} - F - i\alpha\right)$, with the desired result

$$\det^{-1}\left[1-\exp(F+i\alpha)\right] = 1 + e^{i\alpha}\mathrm{tr}\left[\exp(F)\right] + \ldots$$

Thus the problem has nothing to do with the definition of Grassmannian path integration. The problem does not arise, on the other hand, for second order operators or, what is equivalent, for first order operators acting on real fields, such as

$$\det_{PBC}\left(\frac{d}{dt}\delta_{\alpha\beta} - R_{\alpha\beta}\right)$$

the antisymmetry of R here ensures that $\det(e^R) = e^{\mathrm{tr}R} = 1$.

We have thus derived, if not rigourously then consistently, the result of the Atiyah-Singer index theorem for the Dirac and twisted Dirac operators using supersymmetric quantum mechanics and elementary path integral methods. In the process we have come to a better understanding of such methods and, though noting their drawbacks, have confirmed their strength.

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Chapter 4
Grand Unification and Grassmannian Kaluza-Klein Theory

1. Preamble

Kaluza-Klein theories [1] consider the forces of our four-dimensional world - gravity and various Yang-Mills gauge theories - as being contained within the single gravitational force of a higher dimensional space. Recently [2] it was shown that the Einstein-Yang-Mills theory can also be found within a supergravity theory on a (4+N)-dimensional superspace. That is the Kaluza-Klein idea also holds when the extra dimensions are Grassmannian. After reviewing briefly conventional Kaluza-Klein through the simplest (and original) such model, that based on a five-dimensional space-time, and then gravity on a superspace and the Grassmannian Kaluza-Klein ansatz for general Yang-Mills theories, we will consider in this chapter how the standard grand unified theories, the SU(5) and SO(10) models, fit into the Grassmannian Kaluza-Klein framework [3].

2. Conventional Kaluza-Klein Theory

There are two levels at which Kaluza-Klein theory may be considered. At the level of the ansatz the form of the higher dimensional metric is restricted so that its four-dimensional field content is just that of gravity and Yang-Mills theory, or some slight generalization thereof, and so that the higher-dimensional Einstein-Hilbert action reduces to the Einstein-Yang-Mills action on substitution of the metric ansatz and integration over the extra dimensions. A more complete treatment is achieved when a specific form is chosen for the higher-dimensional space-time, such as (locally) $M_4 \times X$, where X is a compact manifold whose length scale is of the order of the Planck length and which has the desired Yang-Mills gauge group as its symmetry group. Then the full higher dimensional metric may be expanded in the extra coordinates with a suitable ansatz for the four-dimensional theory emerging naturally as the low-energy part and with, hopefully, the higher modes not contributing in any adverse manner.

Let us consider the original five-dimensional Kaluza-Klein theory [4]. We will denote the co-ordinates of the five-dimensional space by (x^μ, y), where x^μ co-ordinatize a four-dimensional space-time, and in order to distinguish the five-dimensional metric, etc. from their four-dimensional counterparts we will write these quantities with a ^ above.

The five-dimensional Kaluza-Klein ansatz is

$$\hat{g}_{MN}(z) = \begin{pmatrix} g_{\mu\nu}(x) - \kappa^2 A_\mu(x) A_\nu(x) & -\kappa A_\mu(x) \\ -\kappa A_\nu(x) & -1 \end{pmatrix} \tag{1}$$

or

$$d\hat{s}^2 = g_{\mu\nu}(x) dx^\mu \otimes dx^\nu - (dy + \kappa A_\mu(x) dx^\mu) \otimes (dy + \kappa A_\nu(x) dx^\nu) \tag{2}$$

where $\kappa^2 = 16\pi G$, G being Newton's constant.

This form for the metric is preserved under general co-ordinate transformations of the x^μ if $g_{\mu\nu}(x)$ transforms as a four-dimensional tensor, and under transformations of y of the form

$$y \to y + f(x^\mu)$$

if then

$$A_\mu(x) \to A_\mu(x) - \kappa^{-1}\partial_\mu f(x) ,$$

that is if $A_\mu(x)$ undergoes a U(1) transformation.

It is convenient to work in the horizontal lift basis θ^M of 1-forms,

$$\theta^\mu = dx^\mu, \ \theta^5 = dy + \kappa A_\mu(x)dx^\mu,$$

and the dual basis D_M of tangent vectors

$$D_\mu = \frac{\partial}{\partial x^\mu} - \kappa A_\mu(x)\frac{\partial}{\partial y}, \ D_5 = \frac{\partial}{\partial y}$$

rather than the bases dz^M and $\partial/\partial z^M$. Then [4]

$$\hat{g}_{MN}(z) = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \hat{g}^{MN}(z) = \begin{pmatrix} g^{\mu\nu}(x) & 0 \\ 0 & -1 \end{pmatrix},$$

the Levi-Civita connection is given by

$$\begin{aligned}
&\hat{\Gamma}^\lambda_{\mu\nu}(z) = \Gamma^\lambda_{\mu\nu}(x),\\
&\hat{\Gamma}^5_{\mu\nu}(z) = -\tfrac{1}{2}\kappa F_{\mu\nu}(x), \text{ where } F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x),\\
&\hat{\Gamma}^\mu_{\nu 5}(z) = \tfrac{1}{2}\kappa F^\mu_{\ \nu}(x) = \hat{\Gamma}^\mu_{5\nu}(x),
\end{aligned} \tag{3}$$

with all other components vanishing, and the Ricci scalar is

$$\hat{R}(z) = R(x) - \tfrac{1}{4}\kappa^2 F^{\mu\nu}(x)F_{\mu\nu}(x) .$$

Thus the Einstein-Hilbert action in five-dimensions

$$\frac{1}{\alpha}\int d^5z \sqrt{\hat{g}}\,\hat{R}(z) \tag{4}$$

reduces to the four-dimensional Einstein-Maxwell action

$$\int d^4x \sqrt{-g}\left(\frac{1}{\kappa^2}R(x) - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)\right) \tag{5}$$

provided that the volume of the fifth dimension

$$V \equiv \int dy = \frac{\alpha}{\kappa^2}.$$

The five-dimensional theory can be considered more fully if the space-time is taken to be $M_4 \times S_1$, that is with the fifth coordinate y parametrizing the circle, which has, of course, U(1) as its symmetry group. Then $\hat{g}_{MN}(z)$ may be expanded as a Fourier series in y. The zeroth order of this expansion will be like (1) except that it will also include a scalar field $\varphi(x)$ allowing for the coordinate dependence of g_{55}. The effective four-dimensional theory [5] which results from the substitution of this expansion in the action (4) involves the fields $g_{\mu\nu}(x)$, $A_\mu(x)$ and the Brans-Dicke field $\varphi(x)$ together with an infinite tower of massive spin-two particles with masses $2\pi n/V$ ($n \in Z^+$).

In order to arrive at an effective four-dimensional theory involving matter fields, these must be introduced into the original five-dimensional theory. The action for a scalar $\Phi(z)$

$$\int d^5z \sqrt{\hat{g}}\, \tfrac{1}{2} \hat{g}^{MN} (\partial_M \Phi)^\dagger (\partial_N \Phi) \tag{6}$$

can be reduced by expanding

$$\Phi(z) = \sum_n e^{\frac{2\pi n}{V} y} \varphi_{(n)}(x) \, .$$

Taking g^{MN} to be as given by the ansatz, (6) becomes

$$\int d^4x \sqrt{-g}\, V \left[\tfrac{1}{2} g^{\mu\nu} \left(\left(\partial_\mu - \tfrac{2\pi i n}{V} \kappa A_\mu\right) \varphi_{(n)} \right)^\dagger \left(\partial_\nu - \tfrac{2\pi i n}{V} \kappa A_\nu\right) \varphi_{(n)} - \left(\tfrac{2\pi n}{V}\right)^2 \varphi^\dagger_{(n)} \varphi_{(n)} \right] \tag{7}$$

That is a massless scalar in five dimensions leads to an effetive four-dimensional theory involving a massless scalar not interacting with the electromagnetic field $A_\mu(x)$ together with an infinite tower of massive scalars which do interact with $A_\mu(x)$; the latter's masses and charges are $2\pi n/V$ and $2\pi n\kappa/V$ respectively. Requiring that the quantum of charge be the electronic charge e leads to the value

$$V = \frac{2\pi\kappa}{e}$$

for the "volume" of S_1 the extra space, that is of the order of the Planck length, and thus the masses of the particles in the tower are of the order of the Planck mass.

In order to discuss spinor fields,the vielbeins, $\hat{e}_M{}^A(z)$, and the spin connection, $\hat{\omega}_M{}^A{}_B(z)$, must be used. The viellbeins are such that

$$\hat{g}_{MN}(z) = \eta_{AB} \hat{e}_M^{\,A}(z) \hat{e}_N^{\,B}(z)$$

with the inverse vielbeins, $E^M{}_A(z)$, satisfying

$$\hat{E}_A^{\,M}(z) \hat{e}_M^{\,B}(z) = \delta_A^{\,B} \, , \text{ etc.}$$

and the spin connection is given in terms of the Levi-Civita connection by

$$\hat{\omega}_{MAB}(z) = \hat{e}_{AN} (\partial_M \hat{E}_B^{\,N} + \hat{\Gamma}^N_{ML} \hat{E}_B^{\,L}) \, .$$

In the horizontal lift basis then, with the metric specified by the ansatz,

$$\hat{e}_M^{\,A}(z) = \begin{pmatrix} e_\mu^{\,\alpha}(x) & 0 \\ 0 & 1 \end{pmatrix} \tag{8}$$

where $g_{\mu\nu}(x) = \eta_{\alpha\beta} e_\mu^{\,\alpha} e_\nu^{\,\beta}$, and

$$\begin{aligned} \hat{\omega}_{\mu\alpha\beta} &= \omega_{\mu\alpha\beta} \\ \hat{\omega}_{5\alpha\beta} &= \tfrac{1}{2} \kappa E_\alpha^{\,\mu} F_{\mu\nu} E_\beta^{\,\nu} \\ \hat{\omega}_{\mu\alpha 5} &= \tfrac{1}{2} \kappa E_\alpha^{\,\nu} F_{\nu\mu} = -\hat{\omega}_{\mu 5\alpha} \end{aligned} \tag{9}$$

with all other components vanishing.

The five-dimensional action for a massless spinor $\hat{\psi}(z)$ is

$$\int d^5z\, \hat{e}\, \overline{\hat{\psi}}(z) \gamma^A \hat{E}_A^{\,M}(z) (\partial_M + \hat{\omega}_{MBC}(z) \tfrac{1}{8} [\gamma^B, \gamma^C])\, \hat{\psi}(z) \tag{10}$$

The five-dimensional Dirac matrices, γ^A, can be realized in terms of the four-dimensional ones, γ^α, as

$$\gamma^A = (\gamma^\alpha, i\Gamma)$$

where $\Gamma = \gamma^0\gamma^1\gamma^2\gamma^3$, and with $\hat{\psi}(z)$ then being a four-dimensional Dirac spinor.

Expanding $\hat{\psi}(z)$ in a Fourier series

$$\hat{\psi}(z) = \sum_n e^{\frac{2\pi i n}{V} y}\, \psi_{(n)}(x)\,,$$

and substituting (8) and (9) into the five-dimensional action (10) gives the eventual result

$$\int d^4x\; e \sum_n \overline{\psi}_{(n)} \Big[i\gamma^\alpha E_\alpha^{\ \mu}\Big(\partial_\mu - \tfrac{2\pi i n}{V}\kappa A_\mu + \tfrac{1}{8}\omega_{\mu\beta\gamma}[\gamma^\beta,\gamma^\gamma]\Big) - \tfrac{\kappa}{16} E_\beta^{\ \mu} E_\gamma^{\ \nu} F_{\mu\nu}[\gamma^\beta,\gamma^\gamma]\Gamma - \tfrac{2\pi i n}{V}\Gamma \Big]\psi_{(n)} \tag{11}$$

A global chiral transformation of $\psi_{(n)}(x)$

$$\psi_{(n)}(x) \rightarrow e^{-\frac{i\pi}{4}\Gamma}\, \psi_{(n)}(x)$$

eliminates Γ from (11), leaving the action as that of a massless, uncharged spinor together with, once again, an infinite tower of massive charged spinors. In addition all spinors have a Pauli coupling $\bar{\psi} F_{\mu\nu}[\gamma^\mu,\gamma^\nu]\psi$ to the electromagnetic fieldwith coupling constant $\kappa/16$. Note that none of these spinors is a candidate for the field of an electron, since here, as in the scalar case, they will only possess the electronic charge if their masses are of the order of the Planck mass, and in any case they possess anomalous magnetic moments.

Light fermions are a major problem for realistic Kaluza-Klein theories. They can only emerge through the complicated topology of the extra space as zero modes of the appropriate Dirac operator. However, Witten [6] has shown that this mechanism can never produce a spectrum chirally asymmetric in its couplings to the Yang-Mills fields coming from the Kaluza-Klein mechanism. We discussed this briefly in Chapter 3, Section I.5.

Other problems with Kaluza-Klein theories emerge when they are quantized. Taking the extra dimensions to be of the order of the Planck length leads to enormous Casimir energies and hence to an unacceptable value of the cosmological constant [7]. Also [5],[8] the combined effect of the infinite tower of spin-two particles becomes significant despite their large masses. On the other hand, taking the extra dimensions to be Grassmannian eliminates both of these problems (although, of course, other arise) as then the extra space need not be small in order to be unobservable and the expansion of fields in the extra coordinates is finite.

3. Supergravity

We will now review the construction of supergravity [9] on a real (4+2N)-dimensional superspace in the superbein formalism, preparatory to the consideration of the Grassmannian Kaluza-Klein ansatz itself.

Let the superspace be locally co-ordinatized by $z^M = (x^\mu, \xi^m)$, where $\mu=0,..,3$, $m=1,...,2N$, x^μ are commuting and ξ^m are Grassmannian. Then

$$z^M z^N = [MN] z^N z^M$$

where $[MN] = \begin{cases} -1 & \text{if } M=m, N=n \\ 1 & \text{otherwise} \end{cases}$, $[M] = [MM]$.

On the superspace tangent vectors, T, and covectors, ω, may be constructed. Locally

$$T = T^M(z) \frac{\partial}{\partial z^M}$$

with $\left\{\frac{\partial}{\partial z^M}\right\}$ a basis for the space of tangent vectors, they correspond to left differentiation when acting on a function on superspace, and

$$\omega = dz^M \omega_M(z)$$

where $\{dz^M\}$ is the dual basis to $\left\{\frac{\partial}{\partial z^M}\right\}$, so that

$$\omega(T) = T^M(z)\omega_M(z) = [M]\omega_M(z)T^M(z) \tag{12}$$

In general, the commutativity of any function will be given by the indices which it carries; so that, e.g. , $T^M = (T^\mu, T^m)$ with T^μ commuting and T^m Grassmannian. The order in which indices are contracted is therefore important. The general rule is that indices are contracted down, as in (12), with sign factors introduced should any other indices intervene between the two which are contracted.

From the vectors and covectors, tensors may be formed. A real (super-pseudo-riemannian) metric tensor ds^2, by means of which a correspondence between vectors and covectors is established (indices are lowered), is introduced.

$$ds^2 = dz^M g_{MN}(z)[N]dz^N = dz^N dz^M g_{MN}(z)$$

with

$$g_{MN} = [MN] g_{NM} ,$$

and

$$T_N = T^M g_{MN} .$$

The inverse g^{MN} of g_{MN}, given by

$$g^{MN} g_{NP} = \delta_P^M , \quad g_{PN} g^{NM} = \delta_P^M$$

and satisfying

$$g^{MN} = [M][N][MN] g^{NM} ,$$

is used to raise indices,

$$\omega^M = \omega_N g^{NM} = [M] g^{MN} \omega_N .$$

Note that g^{MN} is not a tensor, rather $[M]g^{MN}$ is, and contraction of its indices with others is an exception to our rule.

Differential p-forms, Ω, can also be defined through the exterior product of co-vectors (1-forms)

$$dz^M \wedge dz^N = -[MN] dz^N \wedge dz^M .$$

$$\Omega = dz^{M_p} \wedge \ldots \wedge dz^{M_1} \Omega_{M_1 \ldots M_p}(z)$$

The exterior derivative mapping p-forms into (p+1)-forms is defined by

$$d\Omega = dz^{M_p} \wedge \ldots \wedge dz^{M_1} \wedge dz^M \frac{\partial}{\partial z^M} \Omega_{M_1 \ldots M_p}(z) .$$

The properties of these on superspace are fairly much those of those on ordinary space, although graded.

The metric tensor ds^2 may be written locally in terms of the flat metric

$$\eta_{AB} = \begin{pmatrix} \eta_{\alpha\beta} & 0 \\ 0 & \eta_{ab} \end{pmatrix} ,$$

where $\eta_{ab} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$, by moving to an orthonormal frame $\{e^A\}$ of 1-forms, called superbeins.

$$\begin{aligned} ds^2 &= e^A e^B \eta_{BA} \\ &= dz^M e_M^A dz^N e_N^B \eta_{BA} \\ &= dz^M dz^N [MN][AN] e_M^A e_N^B \eta_{BA} \end{aligned} \tag{13}$$

so

$$g_{MN} = [AN] e_M^A e_N^B \eta_{BA} .$$

The properties under conjugation of the superbeins must be established from (13) in accordance with the reality of the metric. The following conjugation rules are consistent

$$e^{\alpha *} = e^{\alpha} , \quad e^{a *} = e^{a+N} \equiv e^{\bar{a}} , \quad e^{\bar{a} *} \equiv e^{a+N *} = e^{a} ,$$

where we have used a for 1,...,N and $\bar{a}$ for a+N, and we have now

$$e^A = (e^{\alpha}, e^{a}, e^{\bar{a}})$$

The relation (13) is preserved under OSp(1,3/2N) rotations of the superbeins

$$e^A(z) \rightarrow e^B(z) L_B{}^A(z) ,$$

since OSp(1,3/2N) is the invariance group of η_{AB}.

Tensors under OSp(1,3/2N) may now be considered, and with the superbeins and their inverses we may change world indices M,N,etc. to frame indices A,B,etc. analogously to the conventional case.

In order for the exterior derivatives of frame tensors to also transform tensorially a connection 1-form $\Phi_A{}^B(z)$ must be introduced

$$\Phi_A{}^B(z) = dz^M \Phi_{MA}{}^B(z)$$

and under a frame rotation

$$\Phi_A{}^B \to (L^{-1}\Phi L)_A{}^B - (L^{-1}dL)_A{}^B \tag{15}$$

so that, for example,

$$\begin{aligned} T^A &= de^A + e^B\Phi_B{}^A \\ &\to de^B L_B{}^A + e^B dL_B{}^A + e^C\Phi_C{}^B L_B{}^A - e^B dL_B{}^A \\ &= T^B L_B{}^A \end{aligned} \tag{16}$$

T^A is called the torsion 2-form.

The curvature 2-form $R_A{}^B$ given by

$$R_A{}^B = d\Phi_A{}^B + \Phi_A{}^C\Phi_C{}^B \tag{17}$$

is also a tensor

$$R_A{}^B \to (L^{-1}RL)_A{}^B$$

Conventionally [10], two conditions are imposed upon the connection in order that it may be uniquely secified in terms of the vielbeins. They are metricity, which implies that the spin connection $\omega_\alpha{}^\beta(x)$ takes its values in the Lie algebra of O(1,3)

$$\omega_\alpha{}^\beta\eta_{\beta\gamma} = -\,\omega_\gamma{}^\beta\eta_{\beta\alpha}$$

and vanishing torsion. The spin connection so constrained is called the Levi-Civita spin connection.

For supergravity the condition of metricity leads to the connection Φ_{AB} taking its values in the superalgebra of OSp(1,3/2N), i.e.

$$\Phi_{AB} = -[AB]\Phi_{BA} \tag{18}$$

where $\Phi_{AB} = \Phi_A{}^C\eta_{CB}$.

Fixing the torsion then completely determines the connection in terms of the superbeins (and the torsion) as in conventional gravity. However, neither in space-time supergravity nor for the Kaluza-Klein ansatz is the torsion constrained to vanish. Rather it is given by its value when the connection is zero and the superbeins adopt what is taken as their flat space forms, which might still have a ξ dependence. For the Grassmannian Kaluza-Klein theory the flat space form will be that when the four-dimensional gravitational and Yang-Mills fields, $e_\mu{}^\alpha(x)$ and $A_\mu(x)$, vanish.

Before considering the Grassmannian Kaluza-Klein theory let us just briefly consider how conventional or space-time supergravity fits into the above framework.

Space-time supergravity is based on a (4+4)-dimensional superspace [9],[11], locally $(x^\mu,\theta^m,\bar\theta_{\dot m})$, $m,\dot m=1,2$, and with $\theta^{m*} = \bar\theta^{\dot m}$, $\bar\theta_{\dot m}^{\,*} = \theta_m$.

η_{AB} is taken to be

$$= \begin{pmatrix} \eta_{\alpha\beta} & & \\ & -\varepsilon_{ab} & \\ & & -\varepsilon^{\dot a\dot b} \end{pmatrix} .$$

The admissible co-ordinate transformations are restricted to just the ordinary

$$
\begin{aligned}
x^{\mu} &\to x'^{\mu}(x) \\
\theta^{m} &\to \theta^{m} \\
\bar{\theta}_{\dot{m}} &\to \bar{\theta}_{\dot{m}}
\end{aligned} \tag{19a}
$$

and also local supersymmetry transformations

$$
\begin{aligned}
x^{\mu} &\to x^{\mu} - i(\theta\sigma^{\mu}\bar{\xi}(x) - \xi(x)\sigma^{\mu}\bar{\theta}) \\
\theta^{m} &\to \theta^{m} - \xi^{m}(x) \\
\bar{\theta}_{\dot{m}} &\to \bar{\theta}_{\dot{m}} - \bar{\xi}_{\dot{m}}(x)
\end{aligned} \tag{19b}
$$

The admissible frame rotations are also restricted to those

$$L_B{}^A \in \ OSp(1,3/4)$$

of the form

$$
L_B{}^A(z) = \begin{pmatrix} L_\beta{}^\alpha(x) & & \\ & L_b{}^a(x) & \\ & & L^{\dot{b}}{}_{\dot{a}}(x) \end{pmatrix} \tag{19c}
$$

where then $L_\beta{}^\alpha$, $L_b{}^a$ and $L^{\dot{b}}{}_{\dot{a}}$ are the same Lorentz transformation but in the vector, $(\frac{1}{2},0)$ and $(0,\frac{1}{2}) = \overline{(\frac{1}{2},0)}$ representations respectively.

Finally the flat space metric is taken to be that one such that the supersymmetric differential operators

$$D_A = \left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial \theta^a} + i\sigma^{\mu}_{a\dot{a}}\bar{\theta}^{\dot{a}}\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial \bar{\theta}_{\dot{a}}} + i\theta^a\sigma^{\mu}_{a\dot{b}}\varepsilon^{\dot{b}\dot{a}}\frac{\partial}{\partial x^\mu} \right)$$

are orthonormal, i.e.

$$D_A = E_A{}^M \frac{\partial}{\partial z^M}$$

giving the flat inverse superbeins $E_A{}^M$. From these the flat superbeins follow and hence the constraint on the torsion, namely

$$T^A = (d\theta^m d\bar{\theta}^{\dot{m}} \sigma^{\alpha}_{m\dot{m}}, 0, 0)$$

$$T^{\alpha}_{m\beta} = 2\delta^{\alpha}_{\beta} T_m \quad , \quad T^{\alpha}_{\dot{m}\beta} = 2\delta^{\alpha}_{\beta} T_{\dot{m}}$$

$$T^{p}_{mn} = 2(\delta^{p}_{m} T_n + \delta^{p}_{n} T_m)$$

$$T^{\dot{m}\dot{n}}{}_{\dot{p}} = 2(\delta^{\dot{n}}_{\dot{p}} T^{\dot{m}} + \delta^{\dot{m}}_{\dot{p}} T^{\dot{n}})$$

4. The Grassmannian Kaluza-Klein Ansatz

Let the (intended) Yang-Mills gauge group be G, with an N-dimensional unitary representation R. The crux of the Grassmannian Kaluza-Klein theory is that

$$O(1,3)\times Sp(2N) \subset OSp(1,3/2N)$$

and

$$R(G)\times R(G)^* \subseteq U(N)\times U(N)^* \subset Sp(2N)$$

where by $R(G)\times R^*(G)$ is meant $\{R(g)\times R^*(g);\, g\in G\}$ and similarly for $U(N)\times U^*(N)$. This is easily seen since, if

$$e^\alpha \to e^\beta l_\beta{}^\alpha\,,\quad e^a \to e^b U_b{}^a\,,\quad \bar{e}^{\bar{a}} \to \bar{e}^{\bar{b}}\, U^{*}_{\bar{b}}{}^{\bar{a}}$$

where $l_\beta{}^\alpha \in O(1,3)$, $U_b{}^a \in U(N)$, then

$$\begin{aligned} e^B e^A \eta_{AB} &= e^\beta e^\alpha \eta_{\alpha\beta} + \bar{e}^{\bar{b}} e^a \delta_{ab} - e^b \bar{e}^{\bar{a}} \delta_{ab} \\ &\to e^\beta e^\alpha l_\beta{}^\gamma l_\alpha{}^\delta \eta_{\delta\gamma} + \bar{e}^{\bar{b}} e^a\, U^{*}_{\bar{b}}{}^{\bar{c}}\, U_a{}^d \delta_{dc} - e^b \bar{e}^{\bar{a}} U_b{}^c\, U^{*}_{\bar{a}}{}^{\bar{d}} \delta_{dc} = e^B e^A \eta_{AB} \end{aligned}$$

The situation is thus similar to that of ordinary Kaluza-Klein theory when the Yang-Mills group is the symmetry group of the extra space.

Let the coordinate system of the superspace (x^μ,ξ^m) have the conjugation properties

$$x^{\mu^*} = x^\mu\,,\quad \xi^{m^*} = \xi^{m+N} \equiv \xi^{\bar{m}}\,,\quad \xi^{\bar{m}^*} \equiv \xi^{m+N^*} = \xi^m\,,\quad m = 1,\ldots,N$$

Then the conjugation properties of objects with flat or curved space indices are the same.

The Grassmannian Kaluza-Klein ansatz is [2] (from now on writing superfields with a ^ above to distinguish them from their four-dimensional counterparts)

$$\hat{e}_M{}^A(z) = \begin{pmatrix} e_\mu{}^\alpha(x) & \rho\xi^n A_\mu(x)_n{}^a \\ 0 & \rho\delta_m{}^a \end{pmatrix} \tag{20}$$

or

$$\hat{e}^A = (dx^\mu e_\mu{}^\alpha(x),\ \rho(dx^\nu\, \xi^n A_\nu(x)_n{}^a + d\xi^n \delta_n^a))$$

where $A(x)_n{}^a$ is a matrix in the Lie algebra of $R(G)\times R^*(G)$ and

$$\rho = \exp\left(\frac{c}{\kappa^2}\xi^2\right),$$

with $\xi^2 = \xi^m\xi^n\eta_{nm}$ and c a dimensionless constant. The component fields $e_\mu{}^\alpha(x)$ and $A_\mu(x)$ are of course intended as the four-dimensional gravitational and gauge fields respectively. The purpose of ρ will become clear later.

The class of transformations which preserve the ansatz (20) consists of: general coordinate transformations of the x^μ,

$$x^\mu \to x'^\mu(x)\,,\quad \xi^m \to \xi^m \tag{21}$$

with then

$$e_\mu{}^\alpha(x) \to \frac{\partial x'^\nu}{\partial x^\mu} e_\nu{}^\alpha(x)\,,\ A_\mu(x) \to \frac{\partial x'^\nu}{\partial x^\mu} A_\mu(x)\,;$$

Lorentz transformations given by the frame rotation

$$L_B{}^A = \begin{pmatrix} l_\beta{}^\alpha(x) & 0 \\ 0 & \delta_b{}^a \end{pmatrix} \tag{22}$$

with then

$$e_\mu{}^\alpha(x) \to e_\mu{}^\beta(x) l_\beta{}^\alpha(x)\,;$$

and "gauge transformations" when simultaneously a frame rotation with

$$L_B{}^A = \begin{pmatrix} \delta_\beta{}^\alpha & 0 \\ 0 & U_b{}^a(x) \end{pmatrix} \tag{23a}$$

and a coordinate transformation

$$\xi^m \to \xi^n U_n{}^m(x) \tag{23b}$$

are made, where $U \in R(G) \times R^*(G)$, and provided that then

$$A_\mu(x) \to U^{-1} A_\mu U - U^{-1} \partial_\mu U \tag{24}$$

That is the transformations of the fields $e_\mu{}^\alpha(x)$ and $A_\mu(x)$ are exactly those of the Einstein-Yang-Mills theory.

The situation here with regard to the restricted class of transformations is similar to that of space-time supergravity where the allowable transformations are restricted to (19). On the other hand in space-time supergravity the reduction of the component fields to the graviton, gravitino and certain auxiliary fields is achieved by showing that all other fields can be gauged away, unlike (20) which can only partially be achieved in such a way. In particular, all component fields which would violate the spin-statistics theorem are omitted, and the restricted class of transformations (21), (22), (23) do not contain any supersymmetry transformations.

Just as in the ordinary Kaluza-Klein theory there is a horizontal lift basis in which calculation is simplified, here the basis θ^M is appropriate, with

$$\theta^\mu = dx^\mu\,,\ \theta^m = dx^\nu (\xi A_\nu(x))^m + d\xi^m\,,$$

together with the dual basis D_M,

$$D_M = (\partial_\mu - (\xi A_\mu)^n \partial_n\,, \partial_m)\,.$$

In this basis

$$\hat{e}_M{}^A = \begin{pmatrix} e_\mu{}^\alpha & 0 \\ 0 & \rho\delta_m^a \end{pmatrix},\ \hat{g}_{MN} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & \rho^2\delta_{mn} \end{pmatrix}.$$

The torsion is constrained to

$$T^A = (0, \theta^m \wedge d\rho\ \delta_m^a)$$

which is the value of $d\hat{e}^A$ when $e_\mu{}^\alpha(x) = \delta_\mu^\alpha$ and $A_\mu(x) = 0$. This together with the condition

$$\Phi_{AB} = -[AB]\Phi_{BA}$$

yields the unique solution for the connection

$$\Phi_A{}^B(z) = \begin{pmatrix} dx^\mu \omega_{\mu\alpha}{}^\beta(x) - \rho^2\theta^m(F_\alpha{}^\beta\xi)_m & -dx^\mu(\xi F_{\mu\alpha})^b\rho \\ -dx^\mu(F_\mu{}^\beta\xi)_a\rho & A_a{}^b \end{pmatrix} \tag{25}$$

where $\omega_{\mu\alpha}{}^\beta(x)$ is the Levi-Civita spin connection from $e_\mu{}^\alpha(x)$, and $F_{\mu\nu}(x)$ is the Yang-Mills field strength tensor from $A_\mu(x)$

The curvature tensor

$$\begin{aligned} \hat{R}_A{}^B &= d\Phi_A{}^B + \Phi_A{}^C\Phi_C{}^B \\ &= dz^N \wedge dz^M \hat{R}_{MNA}{}^B \end{aligned}$$

and the Ricci scalar

$$\hat{R} = [B]\hat{E}_B{}^N \hat{E}_A{}^M [AN]\eta^{AC}\hat{R}_{MNC}{}^B$$

can be found from (25). The result is [2]

$$\hat{R} = R + \tfrac{1}{4}\rho^2\xi F_{\mu\nu}F^{\mu\nu}\xi \,.$$

The Einstein-Hilbert action on the superspace is then

$$S = \frac{1}{\alpha}\int d^{2N+4}z\, \hat{e}\,\hat{R} = \frac{1}{\alpha}\int d^4x\, d^{2N}\xi\, e\rho^{-2N}\left[R + \tfrac{1}{4}\rho^2\xi F^2\xi\right] \tag{26}$$

where $\hat{e} = \mathrm{sdet}(\hat{e}_M{}^A)$. The purpose of ρ can now be seen to be the provision of the necessary ξ terms to saturate the ξ-integral. Performing this integral leads to the four-dimensional action

$$S = \int d^4x\, e\left[\frac{1}{\kappa^2}R + \frac{1}{2g^2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})\right] , \tag{27}$$

with $F_{\mu\nu}(x)$ now taking values just in the Lie algebra of R(G), provided that

$$\alpha = 2^{2N}(-1)^{\frac{1}{2}N(N+1)}N^N c^N \kappa^{-2N-2}$$

and

$$c = -\frac{g^2}{4N}\left(\frac{N-1}{N}\right)^{N-1} .$$

Thus the unification of gravity and Yang-Mills theory within a (4+2N)-dimensional supergravity is achieved.

5. Matter Fields

It is straightforward to write down the action for a general OSp(1,3/2N)-scalar superfield $\hat{\varphi}(z)$. It is

$$\int d^{2N+4}z\, \hat{e}\left([M]\hat{g}^{NM}(\partial_M\hat{\varphi})^*\partial_N\hat{\varphi} + \hat{m}^2\hat{\varphi}^*\hat{\varphi} + \hat{\lambda}(\hat{\varphi}^*\hat{\varphi})^2\right)$$

$$= \int d^4x\, d^{2N}\xi\, e\rho^{-2N}\Big(g^{\nu\mu}((\partial_\mu-(\xi A_\mu)^m\partial_m)\hat{\varphi})^*(\partial_\nu-(\xi A_\nu)^n\partial_n)\hat{\varphi} + \rho^{-2}\eta^{mn}(\partial_m\hat{\varphi})^*\partial_n\hat{\varphi}$$
$$+ \hat{m}^2\hat{\varphi}^*\hat{\varphi} + \hat{\lambda}(\hat{\varphi}^*\hat{\varphi})^2\Big) \qquad (28)$$

To analyze this in terms of the component fields of $\hat{\varphi}(z)$, it will be convenient to write

$$\xi_n = \xi^m\eta_{mn}$$

so that

$$\xi_m = -\bar{\xi}^{\bar{m}} \quad \text{and} \quad \xi^{m\,*} = -\xi_m\ ,$$

and to work with (ξ^m,ξ_m) rather than $(\xi^m,\bar{\xi}^{\bar{m}})$.

Under a gauge transformation (23)

$$\xi^m \to \xi^n U_n{}^m(x)\ ,\quad \bar{\xi}^{\bar{m}} \to \bar{\xi}^{\bar{n}} U^*_{\bar{n}}{}^{\bar{m}}(x)$$

with U unitary, $U^{-1\,t} = U^*$. So

$$\xi^m \to \xi^n U_n{}^m \quad \text{and} \quad \xi_m \to \xi_n U^*_n{}^m = \xi_n (U^{-1})^n{}_m\ . \qquad (29)$$

Expanding $\hat{\varphi}(z)$

$$\hat{\varphi}(z) = \varphi(x) + \tfrac{1}{\kappa}\xi^m\varphi_m(x) + \tfrac{1}{\kappa}\xi_n\varphi^n(x) + \ldots + \tfrac{1}{\kappa^{r+s}}\xi_{n_s}\ldots\xi_{n_1}\xi^{m_r}\ldots\xi^{m_1}\varphi_{m_1\ldots m_r}{}^{n_1\ldots n_s}(x) +$$
$$\ldots + \tfrac{1}{\kappa^{2N}}\xi_{n_N}\ldots\xi_{n_1}\xi^{m_N}\ldots\xi^{m_1}\varphi_{m_1\ldots m_N}{}^{n_1\ldots n_N}(x)\ ,$$

with $\varphi_{m_1\ldots m_r}{}^{n_1\ldots n_s}(x)$ antisymmetric with respect to the interchange of any two m_i or any two n_j. Since $\hat{\varphi}(z)$ is a scalar, under a gauge transformation $\varphi_{m_1\ldots m_r}{}^{n_1\ldots n_s}(x)$ must transform in the $\Lambda^rR^*\otimes\Lambda^sR$ representation of G, where Λ^rR^* denotes the anti-symmetrized Kronecker product of r R^*s and similarly for Λ^sR.

When both r and s are non-zero these representations can be reduced by decomposing into traceless and trace parts, e.g.

$$\xi_n\xi^m = (\xi_n\xi^m - \tfrac{1}{2N}\delta^m_n\xi^2) + \tfrac{1}{2N}\delta^m_n\xi^2$$

where

$$\xi^2 = \eta_{mn}\xi^n\xi^m = 2\xi_n\xi^n\ .$$

Also, if R(G) SU(N), then Λ^rR^* is isomorphic to $\Lambda^{N-r}R$ and Λ^sR to $\Lambda^{N-s}R^*$. This equivalence can be seen through contracting with the completely anti-symmetric ε-tensor, $\varepsilon_{m_1\ldots m_N}$ or $\varepsilon^{m_1\ldots m_N}$.

A typical irreducible term in the expansion of $\hat{\varphi}(z)$ is then, for r and s≤$^N/_2$,

$$\frac{1}{\kappa^{r+s+2t}}(\xi^2)^t(\xi_{n_s}\ldots\xi_{n_1}\xi^{m_r}\ldots\xi^{m_1} - \text{traces})\,\varphi_{m_1\ldots m_r}{}^{n_1\ldots n_s}(x)$$

with φ itself being appropriately anti-symmetric and traceless. Substituting such a term into (28) as an ansatz for $\hat{\varphi}(z)$, the ξ_ns and ξ^ms from $\hat{\varphi}$ alone cannot pair to saturate the ξ-integral because of the tracelessness of φ. Instead the ξ^ms from $\hat{\varphi}$ and ξ_ns from $\hat{\varphi}^*$ pair, and vice versa. Thus after the integration $\hat{\varphi}^*\hat{\varphi}$ becomes $\varphi^\dagger\varphi$. Also $A_\mu(x)$, appearing in (28) through $(\xi A_\mu)^m\partial_m$, is automatically made to take its values in the representation carried by φ. So (28) becomes the conventional action

$$\int d^4x\, e\left[g^{\mu\nu}((\partial_\mu - A_\mu)\varphi)^\dagger(\partial_\nu - A_\nu)\varphi + m^2\varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi)^2\right] \tag{30}$$

The m^2 term here incorporates the $\eta^{mn}(\partial_m\varphi)^\dagger\partial_n\varphi$ term of (28) as well as the m^2 term. To the above the proviso should be added that if $r+s+2t > N/2$, then the $(\hat{\varphi}^*\hat{\varphi})^2$ term disappears due to a superfluity of ξ_ns and ξ^ms , and that if $r+s+2t > N$ then similarly the other, quadratic terms vanish.

If a more complete ansatz were taken for $\hat{\varphi}(z)$ involving more than one irreducible term but all with different (r,s), then each term would lead to an action such as (30) but there could also be interaction terms. These terms, however, could only come about through the $\lambda(\hat{\varphi}^*\hat{\varphi})^2$ term of (28) as within the other terms the ξ_ns and ξ^ms could not be properly paired.

For spinor superfields it is not trivial to write down the action in superspace. This is because the spinor representations of the OSp groups are infinite dimensional (a fact that we also discuss in Chapter 6). This can be seen through the Clifford superalgebra

$$\hat{\gamma}^M\hat{\gamma}^N + [MN]\hat{\gamma}^N\hat{\gamma}^M = 2\eta^{MN}$$

The Sp(2N) part of the superalgebra

$$[\hat{\gamma}^m, \hat{\gamma}^n] = 2\eta^{mn}$$

is that of the creation and annihilation operators of N simple harmonic oscillators,

$$\hat{\gamma}^m = \sqrt{2}\,a^m\,,\ \hat{\gamma}^{\bar{m}} = \sqrt{2}\,\bar{a}^m\,.$$

On the other hand, as we have already broken down the group of transformations allowable, it suffices at the level of the ansatz to merely take O(1,3)-spinor superfields and write down an action invariant under all the allowed transformations

$$\int d^{2N+4}z\,\hat{e}\left[\bar{\hat{\psi}}(z)i\gamma^\alpha E^\mu_\alpha(\partial_\mu - (\xi A_\mu)^n\partial_n - \omega_\mu)\hat{\psi}(z) + \hat{m}\bar{\hat{\psi}}(z)\hat{\psi}(z)\right] \tag{31}$$

This action then decomposes into the conventional action for each irreducible component in the expansion of $\hat{\psi}(z)$ as for the scalar case.

6. Grand Unified Theories

Using the foregoing it is now quite staightforward to consider cases of phenomenological interest. The main criterion that we apply is how easily can the required representations of matter fields be generated from the extra coordinates (ξ^m, ξ_m).

The standard model, SU(3)xSU(2)xU(1), has been considered [2]. It was found that a (4+18)-dimensional superspace was required. Of the nine pairs of Grassmannian coordinates, three were required for SU(3), two for SU(2), while four were required for the U(1) hypercharge group. This was because the various hypercharges which different fields carry cannot be generated by products of a single pair of ξs.

The situation for the SU(5) grand unified model is more favourable [3]. In this model [12] the fermions of each generation occur in a 5 and a 10 representation. For example, in the first generation there is

$$\psi_m = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e \\ -\nu_e \end{pmatrix}_L \quad \text{and a} \quad \chi^{mn} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_3^c & -u_2^c & -u_1 & -d_1 \\ -u_3^c & 0 & u_1^c & -u_2 & -d_2 \\ u_2^c & -u_1^c & 0 & -u_3 & -d_3 \\ u_1 & u_2 & u_3 & 0 & -e^c \\ d_1 & d_2 & d_3 & e^c & 0 \end{pmatrix}_L$$

where $_L$ denotes the fact that these are left-handed spinors and superscript c denotes charge conjugation.

The multiplets of scalar fields required for spontaneous symmetry breaking via the Higgs mechanism with the symmetry breaking route

$$SU(5) \rightarrow SU(3)xSU(2)xU(1) \rightarrow SU(3)xU(1)$$

are a 24 and a 5 respectively. The 5 also interacts with the fermions via a Yukawa coupling in order that they can acquire masses without breaking the chiral gauge symmetry of the model. Writing them as tensors based on the fundamental representation of SU(5) these fields are $H^m(x)$ and $\Sigma_n{}^m(x)$ where Σ is traceless.

Extra scalar fields are required with Yukawa couplings to the fermions if the fermion masses are not to be related in an unrealistic manner. The possible representations are given by the form of the Yukawa coupling

$$\overline{\psi_1^c}\psi_2\varphi ,$$

where ψ_1 and ψ_2 are left-handed spinors and φ is the scalar. This term must be overall an SU(5) singlet. Thus the possibilities are

$$\bar{5}x\bar{5} = \overline{10}+\overline{15}$$
$$\bar{5}x10 = 5+\overline{45}$$
$$10x10 = \bar{5}+45+50$$

Of these the 10, the 15 and the 50 are ruled out on phenomenological grounds because they do not have an electically neutral colour singlet component and thus, if they acquired a vacuum expectation value in order to give masses to the fermions at tree level, the they would also break the SU(3) colour group or the U(1) electromagnetism group. The 45 which remains can be written in tensorial form as

$$K_p^{mn}(x)$$

with K being antisymmetric with respect to interchange of m and n and also traceless.

From the above we see that all matter field representations can be generated out of Grassmannian ξ^m and ξ_m transforming in the 5 and $\overline{5}$ representations of SU(5) respectively under gauge transformations. On the (4+10)-dimensional superspace the complete set of superfields required and their ansatzes in order to realize the model in the Grassmannian Kaluza-Klein framework are as follows.

The superbein

$$\hat{e}_M^{\ A}(z) = \begin{pmatrix} e_\mu^{\ \alpha}(x) & \rho(\xi A_\mu)^a \\ 0 & \rho\delta_m^a \end{pmatrix} \tag{32}$$

with

$$\rho = \exp\left(-\frac{g^2}{20\kappa^2}\left(\frac{4}{5}\right)^4 \xi^2\right)$$

and A_μ taking its values in the 5x5 representation of the Lie algebra.
The left-handed spinorial superfields

$$\hat{\psi}(x,\xi) = \frac{1}{\kappa}\xi^m \psi_m(x) + \frac{1}{\kappa^3}\xi^{m_1}\xi^{m_2}\xi^{m_3}\varepsilon_{m_3 m_2 m_1 nm}\,\chi^{mn}(x)$$

and

$$\hat{\chi}(x,\xi) = \frac{1}{\kappa^2}\xi_n\,\xi_m\,\chi^{mn}(x) \tag{33}$$

with $\hat{\psi}$ being a commuting object and $\hat{\chi}$ anticommuting.

The scalar superfields

$$\hat{H}(x,\xi) = \frac{1}{\kappa}\xi_m\,H^m(x)$$

$$\hat{\Sigma}(x,\xi) = \frac{1}{\kappa^2}(\xi_n\,\xi^m - \frac{1}{10}\delta_n^m\,\xi^2)\Sigma_m^n(x) \tag{34}$$

and

$$\hat{K}(x,\xi) = \frac{1}{\kappa^3}(\xi_n\,\xi_m\,\xi^p - \frac{1}{8}\xi^2(\delta_m^p\,\xi_n - \delta_n^p\,\xi_m))K_p^{mn}(x)\,,$$

with $\hat{\Sigma}$ being commuting and $\hat{H}$ and $\hat{K}$ anticommuting.

The superspace action is given by (26)

$$\frac{1}{\alpha}\int d^{14}z\,\hat{e}\hat{R}$$

for the pure Einstein-Yang-Mills part, together with terms of the form (31) for $\hat{\psi}$ and $\hat{\chi}$, terms of the form (28) for H, Σ and K and finally the Yukawa coupling terms

$$\int d^{14}z\,\hat{e}\left[M_u\overline{\hat{\psi}^c}\hat{\chi}\hat{H} + M_d\overline{\hat{\psi}^c}\hat{\chi}\hat{H}^\dagger + \mathcal{M}_u\,\overline{\hat{\psi}^c}\hat{\chi}\hat{K} + \mathcal{M}_d\,\overline{\hat{\psi}^c}\hat{\chi}\hat{K}^\dagger + \text{hermitian conjugates}\right]$$

Note that the purpose of the extra term $\xi\xi\xi\varepsilon\chi$ in the superfield $\hat{\psi}$ is to provide the Yukawa couplings

$$M_u \varepsilon_{m_1 m_2 m_3 m_4 m_5} \overline{\chi^c}^{m_1 m_2} \chi^{m_3 m_4} H^{m_5}$$

and

$$M_u \, \varepsilon_{m_1 m_2 m_3 m_4 m_5} \overline{\chi^c}^{m_1 m_2} \chi^{m_3 m} K_m^{m_4 m_5} .$$

The situation for the other popular grand unified group, SO(10), is not so favourable. In an SO(10) grand unified model the fermions of each generation, now including a right-handed neutrino, are gathered into a 16-dimensional multiplet of left-handed spinors. Now the 16 is (one of) the spinor representations of SO(10) and it cannot be generated from any of the other representations. Thus a Grassmannian Klauza-Klein theory based on SO(10) must have 16 complex ξ^m.

The Yukawa couplings possible for the 16 are given by examining

$$16 \text{x} 16 = (10+126)_s + 120_a$$

All of 10, 126 and 120 contain neutral colour singlet components and so are suitable. However, only the 120 which is the antisymmetric part of 16x16 can be generated from the ξ^m and ξ_m . In order to generate the 10 or the 126 a further 16 complex Grassmannian ζ^m must be introduced. The Grassmannian Kaluza-Klein theory is then based on a (4+64)-dimensional superspace.

Fortunately this then suffices to generate the representations necessary for symmetry breaking, whether via

$$SO(10) \xrightarrow{16} SU(5) \xrightarrow{45} SU(3)xSU(2)xU(1) \xrightarrow{10} SU(3)xU(1)$$

or

$$SO(10) \xrightarrow{54} SU(4)xSU(2)_L xSU(2)_R \xrightarrow{45} SU(3)xSU(2)_L xSU(2)_R xU(1)$$
$$\xrightarrow{16} SU(3)xSU(2)xU(1) \xrightarrow{10} SU(3)xU(1)$$

In conclusion we can say that the SU(5) model is certainly more economical within a Grassmannian Kaluza-Klein framework than an SO(10) one. Of course, the Grassmannian Kaluza-Klein scheme as presented here is just at the level of the ansatz. Any analysis of the full spectrum of the theory would have the advantage over conventional Kaluza-Klein theory of there being only a finite number of modes. On the other hand, half of the modes are unphysical in their spin-statistics, so some mechanism must be found to prevent them from interacting with the physical modes; the question of the spinor representations of OSp(4/2N) also remains.

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Chapter 5
Sp(2)-BRST Quantization

1. Introduction

The BRST supersymmetry is of central importance in covariantly quantized gauge theories. The identities which it implies are necessary for renormalization [1] and the BRST operator is used in the physical state conditions which guarantee unitarity in the operator formalism [2]. Also the cohomology of the BRST operator is of relevance in the study of anomalies [3]. Yet in the conventional Faddeev-Popov approach to the construction of the gauge-fixed action, the BRST invariance only arises incidentally. Consequently a number of other approaches [4]-[8] have been developed which give a more central role to the BRST invariance. Fradkin and others [9] have introduced ghosts and subsequently the BRST invariance into Dirac's framework for the quantization of constrained systems. Methods of quantization based on BRST have recently been applied by many authors to the quantization of string theories [10].

There are further inadequacies of the Faddeev-Popov approach. It can only generate ghosts in pairs (ghost-antighost) and so can only cancel an even number of unphysical degrees of freedom. Also it only generates terms in the quantum lagrangian which are quadratic in the ghost fields. The former is unsatisfactory for antisymmetric tensor fields [11], for example - in four dimensions a second rank antisymmetric tensor field has six degrees of freedom but with the Kalb-Ramond action only one propagates. The latter will not always suffice for renormalization and unitarity [12], quartic ghost terms are possible and sometime also necessary. Another problem can arise if unitarity is being checked by the method of Kugo and Ojima [2], for this the square of the BRST operator must vanish without the use of the equations of motion requiring, in general, the presence of auxiliary fields; these are not given by the Faddeev-Popov technique.

There is often in quantum lagrangians another symmetry, which is like the BRST symmetry with ghosts and anti-ghosts interchanged. It is known as the anti-BRST symmetry [13], and the two together as the extended-BRST symmetry. The anti-BRST symmetry is not actually required for renormalization or unitarity and is not always present. Nevertheless it is remarkable that by requiring that the quantum theory have an extended-BRST symmetry the problems above do not arise.

This requirement is made in a number of schemes [4],[6],[7] usually by formulating the gauge theory on a (4+2)-dimensional superspace coordinatized by $z^M = (x^\mu,\theta,\bar{\theta})$ or (x^μ,θ^m), m=5,6, with translation in the Grassmannian directions giving the extended-BRST transformations of the superfields of the theory. Bosonic fields $A_{\mu_1...}(x)$ of the classical theory with some symmetry under interchange of indices become superfields $A_{M_1...}(x,\theta)$ with the corresponding graded symmetry. This has been the approach used for tensorial fields in previous schemes based on a superspace, effectively it means that O(1,3) representations are replaced by OSp(1,3/2) ones. Constraints are then imposed upon the component field content of the superfields in the "θ-directions" through a curvature superfield or through coset space dimensional reduction. The remaining component fields are then appropriate for the quantum theory corresponding to the original classical theory, with a translation invariant action for the superfields giving an action for the quantum theory which has an extended-BRST invariance.

This method cannot be easily carried over to handle spinor fields as OSp(1,3/2) does not have suitable spinor representations. We give instead a less ambitious approach [14] viewing the superspace more just as a device for the enforcing of the extended-BRST symmetry. By taking superfields only in O(1,3) representations we avoid the need to impose constraints and the problems with spinor fields, while still getting all the results of the more ambitious schemes. We maintain an Sp(2) symmetry in our formulation as this gives a more compact form for the extended-BRST transformations and provides a more general framework for ghosts than the ghost-antighost form. Before considering this we will review the Faddeev-Popov approach for gauge theories.

2. Faddeev-Popov Quantization and BRST

The presence of a local gauge invariance is a problem in the quantization of a classical field theory in that it implies that the operator in the quadratic approximation to the lagrangian for the gauge fields cannot be inverted to give the gauge field's propagator. This reflects the fact that the equations derived from the principle of least action are not all equations of motion, some of them are constraints. In principle, in the non-interacting case these constraint equations could be solved and used to eliminate the non-propagating degrees of freedom of the gauge field. However, the remaining degrees of freedom will not, in general, carry a representation of the Lorentz group so that the manifest covariance of the theory will be lost. In any case, when interactions are present the constraints will not usually be invertible.

The alternative to this approach of restricting the phase space is to add a gauge-fixing term to the lagrangian to break the gauge invariance so that the propagators can be found. The space of physical states corrsponding to incoming, and hopefully outgoing, particles in a scattering process can then be restricted to those which propagated in the gauge invariant theory. This suffices for electromagnetism. However, for self-interacting theories such as non-abelian gauge theories and gravity the presence of negative norms in the space of unphysical states leads to violations of unitarity.

It was realized early [15] that, in the context of Feynman diagrams, unitarity could be restored if extra fictitious fields, called ghosts, were added. These fields by having spin-statistics opposite to those prescibed by the spin-statistics theorem serve to restore unitarity. The understanding was that the enlargement of the phase space to include the unphysical modes of the gauge field must be compensated for by the addition of the same number of ghost degrees of freedom. This idea has been put on a sound footing by Fradkin and others [9]. The arguments above still leave undetermined the ghost lagrangian, unless it is still to be found by trial and error from the Feynman diagrams. Faddeev-Popov [16], working in the path integral approach, resolved this difficulty to some extent. Their approach is briefly described below for a Yang-Mills gauge theory.

Classically, the Yang-Mills theory with a gauge group G is described by the gauge field

$$A_\mu(x) \equiv A_\mu^a(x) t_a$$

taking its values in the Lie algebra of G, with the curvature

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)]$$

and, when the Lie algebra is taken in some representation such that the generators t_a are normalized so that

$$\mathrm{tr}(t_a t_b) = -\frac{1}{2}\delta_{ab} ,$$

with action

$$\int d^4x \, \mathrm{tr}(\tfrac{1}{2} F_{\mu\nu}(x) F^{\mu\nu}(x)) .$$

The action is invariant under the gauge transformations

$$A_\mu(x) \to A_\mu^g(x) = g^{-1}(x) A_\mu(x) g(x) + g^{-1}(x)\partial_\mu g(x) , \qquad (1)$$

where g(x) takes its values in the corresponding representation of G, since then

$$F_{\mu\nu}(x) \to g^{-1}(x) F_{\mu\nu}(x) g(x) .$$

Infinitesimally

$$g(x) = e^{\Lambda(x)} = 1 + \Lambda(x) ,$$

where $\Lambda(x) = \Lambda^a(x) t_a$, and

$$\begin{aligned} A_\mu(x) &\to A_\mu(x) + D_\mu \Lambda(x) \\ &= A_\mu(x) + \partial_\mu \Lambda(x) + [A_\mu(x), \Lambda(x)] \end{aligned} \qquad (2)$$

In the quantized theory the generating functional is given by the path integral

$$Z[J^\mu] = \int' DA_\mu(x) \exp\Big(i\int d^4x \, \mathrm{tr}(\tfrac{1}{2} F_{\mu\nu}(x) F^{\mu\nu}(x) - 2J^\mu(x) A_\mu(x)) \Big) \qquad (3)$$

where the prime denotes that the integration should be carried out only over a space of physically distinct or gauge-inequivalent $A_\mu(x)$ given by some gauge condition. For example, the space of $A_\mu(x)$ given by the covariant condition

$$\partial^\mu A_\mu(x) = \lambda(x) , \qquad (4)$$

for some Lie algebra valued $\lambda(x)$. While $Z[J^\mu]$ is different for different choices of gauge condition, physical scattering amplitudes are not.

The path integral (3) with the condition (4) can be written as an integral over all $A_\mu(x)$

$$Z[J^\mu] = \int DA_\mu(x) \, \delta(\partial^\mu A_\mu(x) - \lambda(x)) \, \Delta(A_\mu(x)) \exp\Big(i\int d^4x \, \mathrm{tr}(\tfrac{1}{2} F^2 - 2J.A) \Big) , \qquad (5)$$

through the insertion of a δ-function and also a $\Delta(A)$, where (3) and (5) together serve to define $\Delta(A)$. It can then be shown that

$$\Delta(A) = \det(\partial^\mu D_\mu)$$

where the operator $\partial\cdot D$ arises from

$$\partial^\mu A^g_\mu(x) - \partial^\mu A_\mu(x) = \partial^\mu D_\mu \Lambda(x),$$

in which $g = \exp(\Lambda(x))$. $\Delta(A)$ is known as the Faddev-Popov determinant.

Since physical quantities are not affected by the choice of $\lambda(x)$, it may be integrated over with a gaussian weight

$$\int D\lambda(x) \exp(i\int d^4x \, \mathrm{tr}(\tfrac{1}{\alpha}\lambda^2))$$

in order to eliminate the δ-function from the expression for $Z[J^\mu]$.

Finally the Faddeev-Popov determinant can be written as the gaussian path integral over a new set of Lie algebra valued fields $\omega(x)$, $\omega(x)$

$$\Delta(A) = \int D\omega(x) D\overline{\omega}(x) \exp\left(i\int d^4x \, \mathrm{tr}(-2\overline{\omega}\,\partial\cdot D\,\omega)\right)$$

where $\omega(x)$ and $\omega(x)$ must be Grassmannian, in order for the integral to give $\Delta(A)$ and not $\Delta^{-1}(A)$, and thus violate the spin-statistics theorem.

(5) now becomes

$$Z[J^\mu] = \int DA_\mu(x) D\omega(x) D\overline{\omega}(x) \exp\left(i\int d^4x \, \mathrm{tr}(\tfrac{1}{2}F^2 + \tfrac{1}{\alpha}(\partial\cdot A)^2 - 2\overline{\omega}\partial^\mu D_\mu\omega - 2J^\mu A_\mu)\right) \quad (6)$$

with the quantum action

$$S[A_\mu(x),\omega(x),\overline{\omega}(x)] = \mathrm{tr}(\tfrac{1}{2}F^2 + \tfrac{1}{\alpha}(\partial\cdot A)^2 - 2\overline{\omega}\partial^\mu D_\mu\omega) \quad (7)$$

containing gauge fixing and ghost terms.

The fact that the quantum action is not itself invariant under gauge transformations considerably complicated the renormalization program until it was observed [1] that the quantum action retains a global symmetry, the BRST symmetry, given exactly by the transformations

$$\begin{aligned} \delta_\varepsilon A_\mu &= \varepsilon D_\mu \omega \\ \delta_\varepsilon \omega &= -\tfrac{1}{2}\varepsilon\{\omega,\omega\} \\ \delta_\varepsilon \overline{\omega} &= -\tfrac{1}{2}\varepsilon\partial_\mu A^\mu, \end{aligned} \quad (8)$$

where ε is an anticommuting parameter. These transformations are nilpotent when the equations of motion are taken into account.

To ensure nilpotency off-shell an auxiliary field $B(x)$ must be introduced.

$$S[A_\mu,\omega,\overline{\omega},B] = \int d^4x \, \mathrm{tr}\left(\tfrac{1}{2}F^2 - \tfrac{1}{\alpha}B^2 + 2B\partial_\mu A^\mu - 2\overline{\omega}\partial^\mu D_\mu\omega\right) \quad (9)$$

is invariant under the transformations given by

$$\delta_\varepsilon = \varepsilon s,$$

$$
\begin{aligned}
&sA_\mu = D_\mu \omega \\
&s\omega = -\tfrac{1}{2}\{\omega,\omega\} \\
&s\overline{\omega} = B \\
&sB = 0
\end{aligned}
\tag{10}
$$

where s, the BRST operator, is linear, graded-Leibniz, commutes with ∂_μ and is nilpotent

$$s^2 = 0\,.$$

For such a theory, invariant under a nilpotent BRST transformation, Kugo and Ojima [2] showed that unitarity could be achieved in the framework of canonical quantization with the space of physical states restricted to a certain subspace of those which are annihilated by the BRST operator.

The anti-BRST symmetry of the action (9) is given by

$$
\begin{aligned}
&\bar{s}A_\mu = D_\mu \overline{\omega} \\
&\bar{s}\omega = -\{\overline{\omega},\overline{\omega}\} - B \\
&\overline{s\omega} = -\tfrac{1}{2}\{\overline{\omega},\overline{\omega}\} \\
&\bar{s}B = -[\overline{\omega},B]
\end{aligned}
\tag{11}
$$

It is also nilpotent,

$$\bar{s}^2 = 0\,.$$

For a different gauge fixing the anti-BRST transformation will take a different form. For example, in the axial gauge, the action

$$S[A_\mu,\omega,\overline{\omega},B] = \int d^4x\ \mathrm{tr}\left(\frac{1}{2}F^2 - \frac{1}{\alpha}B^2 + Bn^\mu A_\mu - 2\overline{\omega}n^\mu D_\mu\omega\right)$$

is invariant under (10) but the anti-BRST transformations are given by

$$
\begin{aligned}
&\bar{s}A_\mu = D_\mu \overline{\omega} \\
&\overline{s\omega} = B \\
&\bar{s}\omega = -\tfrac{1}{2}\{\omega,\omega\} \\
&\bar{s}B = 0
\end{aligned}
$$

Note that quartic ghost terms of the form

$$\int d^4x\ \mathrm{tr}(\{\omega,\omega\})^2$$

are acceptable both on the grounds of renormalizability and of BRST invariance, although not generated in the Faddeev-Popov argument and not actually required for the renormalization of (9).

The Faddeev-Popov result might be summarized using the BRST transformation in the following way. Given a gauge theory with gauge field A(x) and classical action

$$\int d^4x\ L_c(A(x))$$

invariant under the transformation

$$A(x) \to A(x, \Lambda^a(x)) ,$$

the quantum theory consists of the gauge field $A(x)$, ghosts $\omega^a(x)$, antighosts $\overline{\omega}^a(x)$ and auxiliary fields $B^a(x)$. The BRST transformations are given by

$$\begin{aligned} \delta_\varepsilon &= \varepsilon s \\ \delta_\varepsilon A &= A(x;\varepsilon\omega^a) - A(x) \\ s\overline{\omega}^a &= B^a \\ sB^a &= 0 \end{aligned} \tag{12}$$

and

$$s^2 = 0$$

$s\omega^a$ follows from $\delta_\varepsilon A$ and $s^2 = 0$ and the requirement that s be a linear, graded-Leibniz operator commuting with ∂_μ. The BRST-invariant quantum action is

$$\int d^4x \left(L_c(A) + s(\overline{\omega}^a(f_a(A) + \tfrac{1}{2\alpha} B_a)) \right) \tag{13}$$

where $f_a(A)$ are chosen to give a suitable gauge fixing.

Once again, this cannot lead to quartic ghost terms which are necessary for renormalization in non-linear gauges in Yang-Mills theories and unitarity in supergravity [7],[12]. The other inadequacy mentioned in the introduction can be seen for the antisymmetric tensor field:-

In four-dimensions, the second-rank antisymmetric tensor field theory is based around the free action

$$\int d^4x (\tfrac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \partial^\nu A^{\rho\sigma}(x))^2 = \int d^4x \, (\tfrac{1}{2} A^{\mu\nu}(x) \; A_{\mu\nu}(x) + \partial^\mu A_{\mu\nu}(x) \partial_\rho A^{\rho\nu}(x)) \tag{14}$$

which is invariant under

$$A_{\mu\nu}(x) \to A_{\mu\nu}(x) + \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

Applying (12)

$$\begin{aligned} sA_{\mu\nu} &= \partial_\mu \omega_\nu - \partial_\nu \omega_\mu \\ s\overline{\omega}_\mu &= B_\mu \\ sB_\mu &= 0 \end{aligned} \tag{15}$$

and $s\omega_\mu$ must satisfy

$$\partial_\mu s\omega_\nu - \partial_\nu s\omega_\mu = s^2 A_{\mu\nu} = 0 \tag{16}$$

The quantum action (12) is

$$\int d^4x \, (L_c(A) + \overline{\omega}^\mu \partial^\nu (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) + B^\mu \partial^\nu A_{\mu\nu} + \tfrac{1}{2\alpha} B^\mu B_\mu) \tag{17}$$

where we have taken $f_\mu(A) = \partial^\nu A_{\mu\nu}$.

This action has secondary invariances

$$\begin{aligned} \omega_\mu &\to \omega_\mu + \partial_\mu \Lambda \\ \overline{\omega}_\mu &\to \overline{\omega}_\mu + \partial_\mu \overline{\Lambda} \end{aligned} \tag{18}$$

which could be fixed through a further application of (12) and (13), introducing secondary ghosts, now with physical spin statistics, and altering (15) to allow for these. However, introducing two pairs of secondary ghosts for (18) gives overall two

(6-8+4) physical degrees of freedom, while the correct result is one - that is there should only be three secondary ghosts. We will see below that this multiplet can be understood in an Sp(2) framework, rather than in terms of ghosts and antighosts.

3. Sp(2)-BRST

The prescription [14] that we give here as an alternative to the one (12) resulting from the Faddeev-Popov method is based on the requirement that the set of quantum fields carry a representation of the extended-BRST algebra, which is isomorphic to the two-dimensional abelian superalgebra T(2), generated by s_m ($m = 1,2$) with

$$\{s_m, s_n\} = 0 ,$$

the form of the transformations of the original classical fields coming from their gauge transformations and the algebra giving the transformations of the remaining quantum fields. We introduce this algebra by means of a superfield construction based on a superspace coordinatized by (x^μ, θ^m), with metric $\eta_{\mu\nu}$ on the commuting part and ε_{mn} ($\varepsilon_{12} = -1$) on the anticommuting part. The generators of translations in θ^m form the superalgebra T(2), and their action on the superfields we take to be equivalent to the action of the extended-BRST transformations on the component fields. The invariance group of ε_{mn} is Sp(2), so that working covariantly leads to an Sp(2) symmetry between the two BRST transformations, which together we will call the Sp(2)-BRST transformations. In all, the set of quantum fields will carry a representation of $(O(1,3)\mathrm{x}Sp(2))\wedge T(4/2)$.

For a gauge theory as before involving gauge fields $A(x)$ and also matter fields $\psi(x)$, with classical action

$$\int d^4x\, L_c(A(x), \psi(x)) \tag{19}$$

invariant under the transformations

$$\begin{aligned} A(x) &\to A(x;\Lambda^a(x)) \\ \psi(x) &\to \psi(x;\Lambda^a(x)) \end{aligned} \tag{20}$$

we form the superfields $A(x,\theta)$ and $\psi(x,\theta)$ by making super-local gauge transformations

$$\begin{aligned} A(x,\theta) &= A(x;\Lambda^a(x,\theta)) \\ \psi(x,\theta) &= \psi(x,\Lambda^a(x,\theta)) \end{aligned} \tag{21}$$

with

$$\Lambda^a(x,\theta) = \theta^m \omega^a_m(x) + \tfrac{1}{2}\theta^2 b^a(x) \tag{22}$$

where $\theta^2 = \theta^m\theta_m = \varepsilon_{mn}\theta^m\theta^n$. A θ-independent part of $\Lambda^a(x,\theta)$ need not be considered as it may be factored out and absorbed into an ordinary gauge transformation of the classical fields $A(x)$, $\psi(x)$. We have then

$$A(x,\theta)|_{\theta=0} = A(x)$$

$$\psi(x,\theta)|_{\theta=0} = \psi(x)$$

Under an infinitesimal translation

$$\theta^m \to \theta^m + \varepsilon^m$$

a superfield $F(x,\theta)$ has variation

$$\delta_\varepsilon F(x,\theta) = F(x,\theta+\varepsilon) - F(x,\theta)$$

to the first order in ε

$$= \varepsilon^m \frac{\partial}{\partial\theta^m} F(x,\theta) \tag{23}$$

which we interpret as being

$$= \varepsilon^m s_m F(x,\theta) \tag{24}$$

where $\varepsilon^m s_m$ act on the component fields of $F(x,\theta)$ giving their variantions under the Sp(2)-BRST transformations with parameters ε^m. Then

$$F(x,\theta) = F(x) + \theta^m s_m F(x) + \tfrac{1}{2}\theta^2 \tfrac{1}{2}\varepsilon^{mn} s_m s_n F(x) \ . \tag{25}$$

It also follows that

$$\{s_m, s_n\} = 0 \ , \ [s_m, \partial_\mu] = 0$$

and that s_m is linear and satisfies a graded Leibniz rule, i.e.

$$s_m(F(x)G(x)) = (s_m F(x))G(x) \pm F(x)(s_m G(x))$$

according as $F(x)$ is commuting or anticommuting.

Comparing (25) with the superfields $A(x,\theta)$ and $\psi(x,\theta)$ given by (21) immediately gives the expressions for the Sp(2)-BRST transformations of $A(x)$ and $\psi(x)$. It also gives expressions involving the Sp(2)-BRST transformations of $\omega_m{}^a(x)$ and $b^a(x)$. These latter can be solved, with any local freedom in the solution indicative of a secondary gauge invariance and automatically generating new independent fields - the secondary ghosts. This process is continued until the Sp(2)-BRST algebra closes on the space of fields. To each quantum field there will then be a corresponding superfield of the form (25).

The quantum action for the gauge theory may now be formed. It consists of the original classical action (19) together with gauge-fixing and ghost terms. The gauge invariance of (19) ensures its invariance under the Sp(2)-BRST transformations

$$\int d^4x \, L_c(A(x),\psi(x)) = \int d^4x \, L_c(A(x,\theta),\psi(x,\theta))$$

$$= \int d^4x \, L_c(A(x,\theta+\varepsilon),\psi(x,\theta+\varepsilon))$$

The form of the remaining terms must be determined separately for each theory in accordance with the requirements of renormalizability and unitarity. The former requires that the canonical dimension of the terms in the lagrangian density be less than or equal to four without the introduction of dimensionful constants, and the latter that the action be hermitian. The hermiticity and dimensionality of $\omega_m{}^a(x)$, $b^a(x)$, etc.

and of θ^m and s_m must be determined in accordance with these requirements. Of course, there must also be at least one BRST-like invariance of the quantum action.

The full Sp(2)-BRST invariance can be assured for terms of the form

$$\int d^2\theta\ f(A(x,\theta),...) = \frac{1}{2}\varepsilon^{mn}s_m s_n f(A(x),...)$$

due to the invariance of the θ-integral under θ-translations or due to the fact that $\{s_m,s_n\} = 0$. Sp(2)-BRST invariant terms of the form

$$s_m g_n(A(x),\omega_p(x),...)\ ,$$

where g_m is an Sp(2) vector of the appropriate canonical dimension, with must then satisfy

$$\frac{1}{2}s^2 g_m = 0,$$

but with

$$g_m \neq s_m f \ \text{ for any f,}$$

may also be possible, in general. For the cases of the Yang-Mills field, the antisymmetric tensor field and the Rarita-Schwinger field, whose gauge transformations are of the form

$$\delta_\Lambda A(x) = \partial\Lambda + ...$$

and which are considered below, the general form

$$\int d^4x\ \left(L_c(A(x)) + \frac{1}{2}\varepsilon^{mn}s_m s_n(\xi_1\langle A(x),A(x)\rangle + \xi_2\varepsilon^{pq}\langle\omega_p(x),\omega_q(x)\rangle + ...)\right)\ , \qquad (26)$$

where ξ_1, ξ_2 are dimensionless, $\langle A(x),A(x)\rangle$ is an inner product which breaks the gauge invariance and $\langle\omega(x),\omega(x)\rangle$ is a suitable bilinear, gives correct quantum actions. Here the canonical dimensions are

$$[s] = 2 - [A],\ \ [\omega] = 1,\ \ [b] = 2,\ \text{ etc.}$$

and the hermiticity assignments are

$$\omega_1^{a\dagger} = -\,\omega_2^a\ ,\ \ \theta^{1\dagger} = \theta^2\ ,\ \text{ etc.,}$$

order being reversed under hermitian conjugation.

4. Yang-Mills

Applying our prescription to the pure Yang-Mills case, we form the superfield

$$\begin{aligned} A_\mu(x,\theta) &= A_\mu(x;\Lambda(x,\theta)) \\ &= e^{-\Lambda(x,\theta)}A_\mu(x)e^{\Lambda(x,\theta)} + e^{-\Lambda(x,\theta)}\partial_\mu e^{\Lambda(x,\theta)}\ , \end{aligned}$$

where $\Lambda(x,\theta) = \theta^m\omega_m(x) + \frac{1}{2}\theta^2 b(x)$ and $\Lambda = \Lambda^a t_a$, etc.

Then

$$A_\mu(x,\theta) = (1 - \Lambda + \tfrac{1}{2}\Lambda^2)\, A_\mu\, (1 + \Lambda + \tfrac{1}{2}\Lambda^2) + (1 - \Lambda)\, (\partial_\mu\Lambda + \tfrac{1}{2}\partial_\mu\Lambda\Lambda + \tfrac{1}{2}\Lambda\partial_\mu\Lambda)$$
$$= A_\mu(x) + D_\mu\Lambda + \tfrac{1}{2}[D_\mu\Lambda,\Lambda]$$
$$= A_\mu(x) + \theta^m D_\mu\omega_m(x) + \tfrac{1}{2}\theta^2(D_\mu b(x) + \tfrac{1}{2}\varepsilon^{mn}\{D_\mu\omega_m(x),\omega_n(x)\}) \qquad (27)$$

using $\theta^m\theta^n = -\frac{1}{2}\varepsilon^{mn}\theta^2$

Comparing with

$$A_\mu(x,\theta) = A_\mu(x) + \theta^m s_m A_\mu(x) + \tfrac{1}{2}\theta^2\tfrac{1}{2}\varepsilon^{mn}s_m s_n A_\mu(x)$$

we have

$$s_m A_\mu(x) = D_\mu\omega_m(x) \qquad (28)$$

and that

$$\tfrac{1}{2}\varepsilon^{mn}s_m s_n A_\mu(x) = D_\mu b(x) + \tfrac{1}{2}\varepsilon^{mn}\{D_\mu\omega_m(x),\omega_n(x)\} \qquad (29)$$

Now the algebra $\{s_m,s_n\} = 0$ implies that $s_m s_n$ is antisymmetric

$$s_m s_n = \varepsilon_{mn}\tfrac{1}{2}\varepsilon^{pq}s_p s_q \qquad (30)$$

so that

$$s_m s_n A_\mu(x) = \varepsilon_{nm}\tfrac{1}{2}\varepsilon^{pq}s_p s_q A_\mu(x)$$
$$= \varepsilon_{nm}(D_\mu b(x) + \tfrac{1}{2}\varepsilon^{pq}\{D_\mu\omega_p(x),\omega_q(x)\})$$

but also, using (28) and $[s_m,\partial_\mu] = 0$,

$$s_m s_n A_\mu(x) = s_m D_\mu\omega_n(x)$$
$$= s_m\partial_\mu\omega_n(x) + s_m[A_\mu(x),\omega_n(x)]$$
$$= D_\mu(s_m\omega_n(x)) + \{D_\mu\omega_m(x),\omega_n(x)\}$$
$$= D_\mu(s_m\omega_n(x)) + \tfrac{1}{2}D_\mu\{\omega_m(x),\omega_n(x)\} + \tfrac{1}{2}\varepsilon_{nm}\varepsilon^{pq}\{D_\mu\omega_p(x),\omega_q(x)\}$$

where the anticommutator term has been split into symmetric and antisymmetric parts with respect to mn.

Comparison yields

$$s_m\omega_n(x) = \varepsilon_{nm}b(x) - \tfrac{1}{2}\{\omega_m(x),\omega_n(x)\} \qquad (31)$$

To find $s_m b(x)$, note that

$$s_m s_p s_q = 0$$

so that from (29)

$$s_m(D_\mu b(x)) + \tfrac{1}{2}\varepsilon^{np}s_m\{D_\mu\omega_n(x),\omega_p(x)\} = 0$$

and then

$$D_\mu(s_m b(x)) = -[D_\mu\omega_m(x),b(x)] - \tfrac{1}{2}\varepsilon^{np}s_m\{D_\mu\omega_n(x),\omega_p(x)\}$$

which using (28) and (31) eventually yields

$$s_m b(x) = \tfrac{1}{2}[b(x),\omega_m(x)] + \tfrac{1}{12}\varepsilon^{pn}[\{\omega_m(x),\omega_n(x)\},\omega_p(x)] \qquad (32)$$

The quantum action taken in the form of (26) is

$$\int d^4x\ \mathrm{tr}\left[\frac{1}{2}F^2 + \frac{1}{2}\varepsilon^{mn}s_m s_n(\xi_1 A^\mu(x)A_\mu(x) + \xi_2\varepsilon^{pq}\omega_q(x)\omega_p(x))\right]$$

$$= \int d^4x\ \mathrm{tr}\left[\frac{1}{2}F^2 + \xi_1(A^\mu(x)\varepsilon^{mn}s_m s_n A_\mu(x) + \varepsilon^{mn}s_m A_\mu(x)s_n A_\mu(x)) + \xi_2(\varepsilon^{pq}\omega_q(x)\varepsilon^{mn}s_m s_n\omega_p(x) + \varepsilon^{pq}\varepsilon^{mn}s_n\omega_q(x)s_m\omega_p(x))\right]$$

$$= \int d^4x\ \mathrm{tr}\left[\frac{1}{2}F^2 + \xi_1(2A^\mu(x)\partial_\mu b(x) + \varepsilon^{mn}D_\mu\omega_m(x)\partial^\mu\omega_n(x)) + \xi_2(2b^2(x) - \frac{1}{6}\varepsilon^{mn}\varepsilon^{pq}\{\omega_m(x),\omega_p(x)\}\{\omega_n(x),\omega_q(x)\})\right] \qquad (33)$$

The Sp(2)-BRST transformations become the familiar BRST and anti-BRST transformations (10) and (11) if we write

$$\omega_m(x) = (\omega(x),\overline{\omega}(x)),\ \ s_m = (s,\overline{s})$$

and

$$b(x) = B(x) + \frac{1}{2}\{\omega(x),\overline{\omega}(x)\}$$

In terms of these fields (33) is

$$\int d^4x\ \mathrm{tr}\left[\frac{1}{2}F^2 + \xi_1(-2B\partial_\mu A^\mu + 2\overline{\omega}\partial^\mu D_\mu\omega) + \xi_2(2B^2 + 2B\{\overline{\omega},\overline{\omega}\} + \frac{4}{3}(\{\omega,\overline{\omega}\})^2 - \frac{1}{3}\{\omega,\omega\}\{\overline{\omega},\overline{\omega}\})\right] \qquad (34)$$

which for $\xi_1 = -1,\ \xi_2 = 0$ is just (9) in the Landau gauge $\alpha \to \infty$. The extra terms which accompany B^2 are aconsequence of the Sp(2) invariance of our lagrangian. The general form of (33) and (34) is equivalent to those arrived at in [5],[6], where has been shown to lead to the correct quantum theory.

Our scheme is different from these other schemes in that we do not see the superspace (x^μ,θ^m) as anything more than a device for the imposition of the extended-BRST symmetry. That is we do not attempt a unified treatment of the commuting and anticommuting directions. The other schemes by contrast write down a superspace version of Yang-Mills theory involving a superfield

$$A_M(x,\theta) = (A_\mu(x,\theta),A_m(x,\theta))$$

By imposing the condition that admissible $A_M(x,\theta)$ are just $(A_\mu(x),0)$ up to a super-local gauge transformation, they arrive at

$$A_M(x,\theta) = (A_\mu(x;\Lambda(x,\theta)),\omega_m(x)+\theta_m b(x)) \qquad (35)$$

The superfield $A_\mu(x,\theta)$ is now just the same as ours.

No real unification is achieved in these schemes because their own condition (35) denies any equivalence between the x^μ and θ^m directions - it is not preserved under a general OSp(4/2) transformation. As well, that approach cannot easily be applied to spinorial matter fields. Generalizing spinors to the superspace would require them to carry a representation of the Clifford superalgebra

$$[\gamma^M,\gamma^N]_\pm = 2\eta^{MN} = \begin{pmatrix} \eta^{\mu\nu} & 0 \\ 0 & \varepsilon^{mn} \end{pmatrix},$$

but the sector

$$[\gamma^m,\gamma^n] = 2\varepsilon^{mn}$$

has only infinite dimensional representations (as we pointed out in Chapter 4, this is just the simple harmonic oscillator $[a,\bar{a}] = 1$) - related to the fact that OSp(4/2) has no finite dimensional spinor representations of non-vanishing superdimension [6],[17].

For us the incorporation of matter fields $\psi(x)$, transforming under gauge trans-formations as

$$\psi(x) \to e^{-\Lambda^a(x)t_a}\psi(x)$$

in some representation of the t_a, is straightforward whether they are Lorentz scalars or spinors. We form the superfield

$$\begin{aligned}\psi(x,\theta) &= e^{-\Lambda(x,\theta)}\psi(x)\\ &= \psi(x) - \Lambda(x,\theta)\psi(x) + \tfrac{1}{2}\Lambda^2(x,\theta)\psi(x)\\ &= \psi(x) - \theta^m\omega_m(x) + \tfrac{1}{2}\theta^2(\tfrac{1}{2}\varepsilon^{mn}\omega_m(x)\omega_n(x) - b(x))\psi(x)\end{aligned}$$

and thus find that the Sp(2)-BRST transformation of $\psi(x)$ is

$$s_m\psi(x) = -\,\omega_m(x)\psi(x) \tag{36}$$

This together with (28)

$$s_m A_\mu(x) = D_\mu\omega_m(x)$$

gives an invariance of the gauge-invariant matter lagrangian.

5. Anti-symmetric Tensor

From the antisymmetric tensor field $A_{\mu\nu}(x)$ with classical action (14) we form the superfield

$$A_{\mu\nu}(x,\theta) = A_{\mu\nu}(x) + \partial_\mu\Lambda_\nu(x,\theta) - \partial_\nu\Lambda_\mu(x,\theta)$$

where $\Lambda_\mu(x,\theta) = \theta^m\omega_{m\mu}(x) + \tfrac{1}{2}\theta^2 b_\mu(x)$

Thus finding that

$$s_m A_{\mu\nu}(x) = \partial_\mu\omega_{m\nu}(x) - \partial_\nu\omega_{m\mu}(x) \tag{37}$$

$$s_m(\partial_\mu\omega_{n\nu}(x) - \partial_\nu\omega_{n\mu}(x)) = \varepsilon_{nm}(\partial_\mu b_\nu(x) - \partial_\nu b_\mu(x))$$

and

$$s_m(\partial_\mu b_\nu(x) - \partial_\nu b_\mu(x)) = 0$$

Solving, we generate the fields $\omega_{mn}(x)$ and $b_m(x)$.

$$s_m\omega_{n\mu}(x) = \varepsilon_{nm}b_\mu(x) + \partial_\mu\omega_{mn}(x) \tag{38}$$

$$s_m b_\mu(x) = \partial_\mu b_m(x) \tag{39}$$

Any antisymmetric part of $\omega_{mn}(x)$ may be absorbed into $\varepsilon_{mn}b_\mu(x)$ so that we take it to be symmetric. To find the Sp(2)-BRST transformations of $\omega_{mn}(x)$ and $b_m(x)$ we use the antisymmetry of $s_p s_m$.

$$s_p s_m \omega_{n\mu}(x) = \varepsilon_{nm}\partial_\mu b_p(x) + \partial_\mu s_p \omega_{mn}(x)$$

which must be, for some $d_n(x)$,

$$= 2\varepsilon_{pm}\partial_\mu d_n(x)$$

Comparing parts antisymmetric or symmetric in nm, we have

$$d_p(x) = b_p(x)$$

and

$$s_p\omega_{mn}(x) = \varepsilon_{pm}b_n(x) + \varepsilon_{pn}b_m(x) \tag{40}$$

Finally,

$$s_q s_p s_m \omega_{n\mu}(x) = 0\ ,$$

since it is completely antisymmetric in qpm, implying that

$$s_q b_p(x) = 0\ .$$

The quantum action in the form (26) is

$$\int d^4x \left[L_c(A(x)) + \frac{1}{2}\varepsilon^{mn}s_m s_n(\xi_1 A^{\mu\nu}(x)A_{\mu\nu}(x) + \xi_2\omega^{m\mu}(x)\omega_{m\mu}(x) + \xi_3\omega^{mn}(x)\omega_{mn}(x)) \right]$$

which is

$$\begin{aligned}\int d^4x \Big[& L_c(A(x)) + 4\xi_1 b^\mu(x)\partial^\nu A_{\mu\nu}(x) + 2\xi_1\partial^\mu\omega^{m\nu}(x)(\partial_\nu\omega_{m\mu}(x) - \partial_\mu\omega_{m\nu}(x)) \\ & + 2\xi_2 b^\mu(x)b_\mu(x) + \xi_2\partial^\mu\omega^{mn}(x)\partial_\mu\omega_{mn}(x) - 6\xi_3 b^m(x)b_m(x) \Big]\end{aligned} \tag{41}$$

Here $A_{\mu\nu}(x)$, $\omega_{m\mu}(x)$ and $\omega_{mn}(x)$ propagate, with the physical degrees of freedom being $6 - 8 + 3 = 1$ as it should, with the correct number (three) of secondary ghosts coming in through a representation of Sp(2).

6. Rarita-Schwinger

This case has been treated [18] in an OSp(4/2) framework using an infinite dimensional representation of the Clifford superalgebra. However, it was found that a correct result could not be attained. On the other hand, our method although simple-minded does handle this case satisfactorily.

Here the gauge field is a spinor-vector $\psi_\mu(x)$ with classical action

$$\begin{aligned}\int d^4x\, L_c(\psi_\mu(x)) &= \int d^4x\, i\bar{\psi}_\mu(x)\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}\partial_\nu\psi_\rho(x) \\ &= \int d^4x\, i\bar{\psi}_\mu(x)(\eta^{\mu\nu}\not{\partial} - (\partial^\mu\gamma^\nu + \partial^\nu\gamma^\mu - \gamma^\mu\not{\partial}\gamma^\nu))\psi_\nu(x)\end{aligned}$$

invariant under the gauge transformation

$$\psi_\mu(x) \to \psi_\mu(x) + i\partial_\mu\Lambda(x)\ ,$$

with $\Lambda(x)$ a spinor.

By taking $\Lambda(x,\theta) = \theta^m c_m(x) + \frac{1}{2}\theta^2 B(x)$, we form the superfield

$$\begin{aligned}\psi_\mu(x,\theta) &= \psi_\mu(x) + i\partial_\mu\Lambda(x,\theta) \\ &= \psi_\mu(x) + \theta^m i\partial_\mu c_m(x) + \tfrac{1}{2}\theta^2 i\partial_\mu B(x)\end{aligned} \tag{42}$$

Then we have

$$\begin{aligned}s_m\psi_\mu(x) &= i\partial_\mu c_m(x) \\ s_n c_m(x) &= \varepsilon_{mn}B(x) \\ s_n B(x) &= 0\end{aligned} \tag{43}$$

We take

$$\langle\psi(x),\psi(x)\rangle = \overline{\psi}_\mu(x)\gamma^\mu\gamma^\nu\psi_\nu(x)$$

and

$$\langle c_m(x),c_n(x)\rangle = \bar{c}_m(x)\not\partial c_n(x)$$

where ∂ is required to ensure that this term has the correct dimension, then the form (26) gives as the quantum action

$$\int d^4x \left[L_c(\psi_\mu(x)) + \tfrac{1}{2}\varepsilon^{mn}s_n s_m(\xi_1\overline{\psi}_\mu(x)\gamma^\mu\gamma^\nu\psi_\nu(x) + \xi_2\varepsilon^{pq}\bar{c}_q(x)\not\partial c_p(x))\right]$$

$$\begin{aligned}= \int d^4x \Big[L_c(\psi_\mu(x)) &+ \xi_1(\varepsilon^{mn}\bar{c}_m(x)\ \ c_n(x) + i\overline{\psi}_\mu(x)\gamma^\mu\not\partial B(x) - i\overline{B}(x)\not\partial\gamma^\mu\psi_\mu(x)) \\ &+ \xi_2\overline{B}(x)\not\partial B(x)\Big]\end{aligned}$$

This is the quadratic part of the lagrangian that is shown in[19] to lead to a unitary quantum theory for the gravitino part of supergravity.

Conclusion

In this chapter we have discussed a simple prescription for quantizing gauge theories based upon two BRST symmetries related by an Sp(2) transformation. The Sp(2) symmetry leads to a more compact description of the extended-BRST symmetry than does the BRST-anti-BRST approach and provides some rationale for the occurrence of ghosts other than in ghost-antighost pairs. By not attempting to unify the BRST with the space-time symmetries the problems which a method based on OSp(4/2) encounters with spinorial fields have been avoided.

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Chapter 6
Massive Yang-Mills Theory: Renormalizability vs Unitarity

1. Preamble

The Higgs mechanism is the standard device by which massive non-abelian gauge fields are considered within a renormalizable and unitary theory. Yet, until and unless a Higgs particle is observed experimentally, this method must always be open to some doubt. The search for such a particle is of course made difficult by the fact that the Higgs method does not of itself determine the particle's mass nor its coupling constants in terms of other known constants - although constraints upon the values possible have been obtained indirectly. Another problem with the Higgs mechanism arises within the context of grand unified theories - the so-called hierarchy problem.

Recently another approach was suggested [1] as a possible alternative to the Higgs method. It is based upon the non-abelian generalization of the Stueckelberg model. Unfortunately, it was soon discovered [2],[3] that although this approach leads to a renormalizable theory it is only so at the expense of unitarity - whereas ordinary massive non-abelian vector theory might be thought of as unitary but unrenormalizable. We shall in this chapter demonstrate the conflicting nature of the two requirements.

2. Renormalizablity of Massive Gauge Theories and the Stueckelberg Model

We begin with a brief review of massive electrodynamics. This theory is renormalizable even though naive power counting would lead one to the opposite conclusion. The massive Lagrangian, without matter fields, is

$$L_g = -\tfrac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + \tfrac{m^2}{2}A_\mu(x)A^\mu(x) \tag{1}$$

where $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$.

The mass term is not invariant under the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x)$$

and thus the propagator may be found straightaway. It is

$$\frac{\eta_{\mu\nu} - \dfrac{k_\mu k_\nu}{m^2}}{k^2 - m^2 + i\varepsilon} \tag{2}$$

and is of order 1 at high momenta, leading to the expectation that when interactions are added the result will be unrenormalizable by power counting.

This is, however, not the case if the coupling is to a conserved current [4], such as in spinor electrodynamics where

$$L = L(A_\mu(x)) + gj^\mu(x)A_\mu(x) + L_m(\psi(x),\overline{\psi}(x)) \tag{3}$$

$$j^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x)$$

$$L_m(\psi(x),\overline{\psi}(x)) = i\overline{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - M\overline{\psi}(x)\psi(x)$$

and for which

$$gj^{\mu}(x)A_{\mu}(x) + L_m(\psi(x),\overline{\psi}(x))$$

is invariant under a gauge transformation,

$$A_{\mu}(x) \to A_{\mu}(x) + \partial_{\mu}\Lambda(x)$$
$$\psi(x) \to e^{ig\Lambda(x)}\psi(x)$$
$$\overline{\psi}(x) \to e^{-ig\Lambda(x)}\overline{\psi}(x)\ .$$

To see that such a theory is renormalizable we use the Stueckelberg formulation which restores gauge invariance. It is obtained by introducing the Stueckelberg field $\varphi(x)$ through the gauge transformation

$$A_{\mu}(x) \to A_{\mu}(x) - \tfrac{1}{m}\partial_{\mu}\varphi(x)$$

giving the Lagrangian

$$L = -\tfrac{1}{4}F^2 + \tfrac{m^2}{2}(A_{\mu} - \tfrac{1}{m}\partial_{\mu}\varphi)^2 + gj^{\mu}A_{\mu} + L_m \tag{4}$$

which is invariant under

$$\delta A_{\mu} = \partial_{\mu}\Lambda$$
$$\delta\varphi = m\Lambda$$
$$\psi \to e^{ig\Lambda}\psi$$
$$\overline{\psi} \to e^{ig\Lambda}\overline{\psi}$$

and so must be gauge fixed before propagators can be found.

Consider first of all the gauge fixing given by the function $f(A) = \partial\cdot A$. To (4) is added

$$L_{gf} = s(\overline{\omega}(\partial\cdot A + \tfrac{1}{2\alpha}B))$$

where s is the BRST transformation

$$s\varphi = m\omega$$

$$sA_{\mu} = \partial_{\mu}\omega\ , \text{etc.}$$

and

$$s\overline{\omega} = B$$
$$s\omega = 0 \tag{5}$$
$$sB = 0\ ,$$
$$s^2 = 0\ ;$$

i.e.

$$L_{gf} = B\partial.A + \overline{\omega}\Box\omega + \tfrac{1}{2\alpha}B^2\ .$$

The propagator for $A_{\mu}(x)$ may now be found and it is just the standard propagator

$$\frac{\eta_{\mu\nu} - \frac{1-\alpha}{\alpha}\frac{k_\mu k_\nu}{k^2 - m^2/\alpha + i\varepsilon}}{k^2 - m^2 + i\varepsilon}$$

which behaves as $1/k^2$ for large k, thus ensuring renormalizability.

Another gauge choice that may be considered is

$$\begin{aligned} L_{gf} &= s(\overline{\omega}(\varphi + \tfrac{1}{2\alpha}B)) \\ &= B\varphi + m\overline{\omega}\omega + \tfrac{1}{2\alpha}B^2 \end{aligned}$$

Since the lagrangians for the two cases differ only by a BRST invariant term the S matrices which each give will coincide [5]; but for the latter, as $\alpha \to \infty$, $\varphi = 0$ is enforced and the theory is manifestly the original one (4). Thus this theory must be renormalizable.

The non-abelian case is, however, not so favourable. Here the massive lagrangian is

$$\begin{aligned} L &= \tfrac{1}{2}\mathrm{tr}\, F^2 - m^2 \mathrm{tr}\, A^2 \\ &= -\tfrac{1}{4}(F^a_{\mu\nu})^2 + \tfrac{m^2}{2}(A^a_\mu)^2 \end{aligned} \tag{6}$$

where

$$\begin{gathered} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu] \\ A_\mu = A^a_\mu t_a,\ F_{\mu\nu} = F^a_{\mu\nu} t^a \\ [t_a, t_b] = f_{ab}{}^c t_c,\ \mathrm{tr}(t_a t_b) = -\tfrac{1}{2}\delta_{ab} \end{gathered}$$

The mass term breaks the invariance under the gauge transformation

$$A_\mu \to S^{-1}A_\mu S + \tfrac{1}{g}S^{-1}\partial_\mu S$$

where

$$S = \exp(\Lambda^a(x)t_a),$$

and the propagator for $A_\mu(x)$ once again has bad high energy behaviour.

The non-abelian generalization of the Stuckelberg formulation [6] comes about through

$$A_\mu \to UA_\mu U^{-1} + \tfrac{1}{g}U\partial_\mu U^{-1}$$

where

$$U = \exp\left(\tfrac{g}{m}\varphi(x)\right) = \exp\left(\tfrac{g}{m}\varphi^a(x)t_a\right)$$

This leads to the lagrangian

$$L = \tfrac{1}{2}\mathrm{tr}\, F^2 - m^2\, \mathrm{tr}\left[(A_\mu - \tfrac{1}{g}U^{-1}\partial_\mu U)^2\right] \tag{7}$$

$$= \tfrac{1}{2}\mathrm{tr}\, F^2 - m^2 \mathrm{tr}\, A^2 + 2m\, \mathrm{tr}(A^\mu \partial_\mu \varphi) - \mathrm{tr}(\partial^\mu \varphi D_\mu \varphi) + O(g^2) \tag{8}$$

which is invariant under the gauge transformation

$$A_\mu \to S^{-1}\partial_\mu S + S^{-1}\partial_\mu S$$

$$U \to US$$

Once again the gauge may be fixed through the addition of

$$L_{gf} = -2\,\mathrm{tr}\left(s(\overline{\omega}(\partial\cdot A + \tfrac{1}{2\alpha}B))\right)$$

$$= -2\,\mathrm{tr}(B\partial\cdot A + \overline{\omega}\partial.D\omega + \tfrac{1}{2\alpha}B^2) \tag{9}$$

leading to a well-behaved propagator for $A_\mu(x)$. In this case though, renormalizability is not assured because of the non-polynomiality of the Lagrangian. There are an infinite number of monomial interaction terms and it is not clear whether each has to be separately renormalized.

It was, however, observed [1] that the equation of motion for φ is

$$\Box\varphi - m\partial.A + \tfrac{1}{2}g([A^\mu,\partial_\mu\varphi] + \partial_\mu[A^\mu,\varphi]) + O(g^2) = 0$$

or

$$D_\mu(U^{-1}\partial^\mu U) = g\partial\cdot A$$

and that in the Landau gauge $\alpha \to \infty$, when $\partial\cdot A = 0$, $\varphi = 0$ is a solution of this equation. Thus it was proposed that in the Landau gauge this could be imposed leading to the Lagrangian

$$L = \tfrac{1}{4}F^2 + \tfrac{1}{2}mA^2 + B\partial\cdot A + \overline{\omega}\partial\cdot D\omega \tag{10}$$

which is polynomial and gives a propagator for $A_\mu(x)$ with the appropriate high-energy behaviour. In other words, it was suggested that U or the φ field could be eliminated in (7) or (8) in favour of $A_\mu(x)$ through its equation of motion. Then in the Landau gauge the resultant lagrangian with gauge fixing would be just (10).

3. Unitarity

The Lagrangian just obtained (10) does not lead to a unitary theory [2]. It may be observed that the modified BRST transformations under which (10) is invariant

$$s'A_\mu = D_\mu\omega$$

$$s'\overline{\omega} = B$$

$$s'\omega = -\tfrac{1}{2}g\{\omega,\omega\}$$

$$s'B = m^2\omega$$

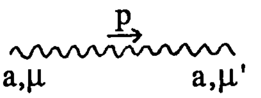
are not nilpotent,

$$s'^2\overline{\omega} = m^2\omega\,,$$

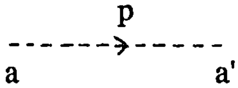
This in itself is perhaps not sufficient to prove that unitarity is violated. Indeed a formal argument for unitarity was given. It was based on a proof of the unitarity of the original non-polynomial theory, which is invariant under a nilpotent BRST transformation, followed by manipulations of the path integral generating function so as to arrive at the theory given by the lagrangian (10). Explicit diagrammatic methods are therefore perhaps the best way to demonstrate the failure of unitarity [2],[3].

The Feyman diagram rules from (9) are:

propagators

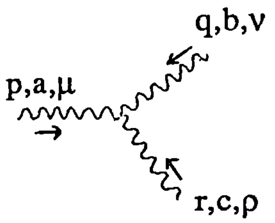


$$\Delta^{aa'}_{\mu\mu'}(p) = \frac{-\eta_{\mu\mu'} + \dfrac{p_\mu p_{\mu'}}{p^2 + i\varepsilon}}{p^2 - m^2 + i\varepsilon}\, i\delta^{aa'}$$

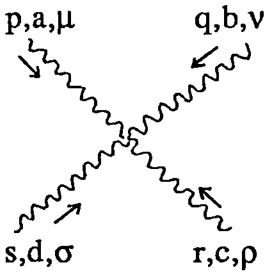


$$D^{aa'}(p) = \frac{1}{p^2 + i\varepsilon}\, i\delta^{aa'}$$

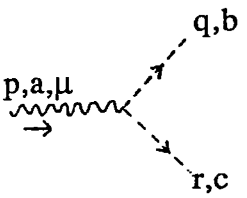
vertices



$$g\Gamma^{\mu\nu\rho}_{abc}(p,q,r) = gf_{abc}((q-r)_\mu\eta_{\nu\rho} + (r-p)_\nu\eta_{\rho\mu} + (p-q)_\rho\eta_{\mu\nu})$$



$$\begin{aligned} g^2C^{\mu\nu\rho\sigma}_{abcd} = -ig^2(&f_{abe}f_{cde}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\ &+ f_{ace}f_{dbe}(\eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}) \\ &+ f_{ade}f_{bce}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})) \end{aligned}$$



$$g\Gamma^{\mu}_{abc}(q) = gf_{abc}q^\mu$$

with $(2\pi)^4\delta^4(\sum_i p_i)$ for the incoming momenta at each vertex,

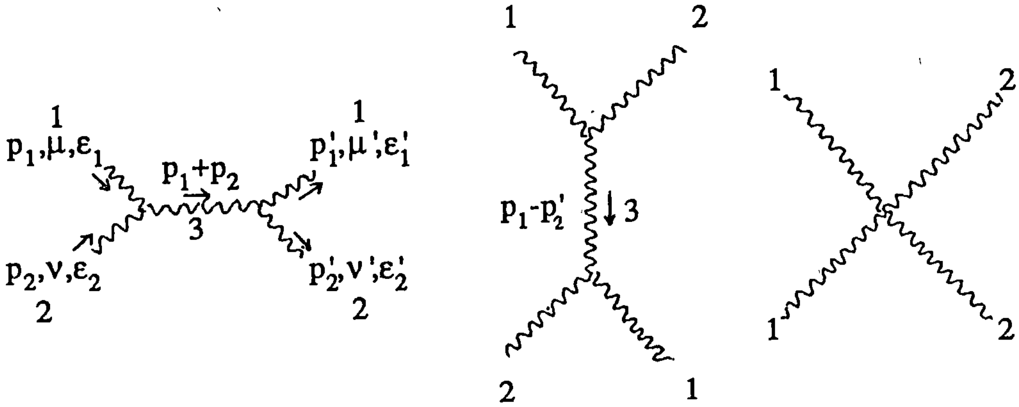
$\int \frac{d^4p_i}{(2\pi)^4}$ for each internal momentum,

a factor of (-1) for each ghost loop, and a symmetry factor $1/S$ for any symmetry of the internal lines of a graph.

1. The first process which we shall examine is the elastic scattering of two longitudinally polarized spin-1 states. For simplicity, and so that we may compare the results with those for the Higgs mechanism, we will work with an SU(2) gauge

theory, so that $a = 1,2,3$, $f_{abc} = \varepsilon_{abc}$.

There are unitarity bounds for the scattering of two spin-1 particles [7],[8] and in order that they be satisfied the amplitude must at each order in g^2 be of order 1 at high energies. Consider $A^1{}_L A^2{}_L \to A^1{}_L A^2{}_L$. Now at tree level, $O(g^2)$, the possible diagrams are



yielding contributions T_s, T_t, T_4 respectively to the amplitude T_{tot}.

In the centre of mass frame

$$p_1 = (E,0,0,p),\ \ p_2 = (E,0,0,-p),$$
$$p_1' = (E,p\sin\theta,0,p\cos\theta),\ \ p_2' = (E,-p\sin\theta,0,p\cos\theta)$$
$$\text{with } E^2 - p^2 = m^2,$$

and the longitudinal polarization vectors are

$$\varepsilon_1 = \frac{1}{m}(p,0,0,E),\ \ \varepsilon_2 = \frac{1}{m}(p,0,0,-E)$$
$$\varepsilon_1' = \frac{1}{m}(p,E\sin\theta,0,E\cos\theta),\ \ \varepsilon_2' = \frac{1}{m}(p,-E\sin\theta,0,-E\cos\theta)$$
$$\text{with } \varepsilon_i^2 = -1,\ \ p_i\cdot\varepsilon_i = 0\,.$$

We then have

$$T_s = g^2\varepsilon_{1\mu}\varepsilon_{2\nu}\Gamma^{\mu\nu\rho}(p_1,p_2,-p_1-p_2)\Delta_{\rho\sigma}(p_1+p_2)\Gamma^{\mu'\nu'\sigma}(-p_1',-p_2',p_1'+p_2')\,\varepsilon'_{1\mu'}\,\varepsilon'_{2\nu'}$$
$$= 4g^2\cos\theta\,\frac{(E^2-m^2)(2E^2+m^2)^2}{(4E^2-m^2)m^4}$$
$$= g^2\cos\theta\left(4\left(\frac{E}{m}\right)^4 + \left(\frac{E}{m}\right)^2 + O(1)\right)$$

$$T_u = g^2\varepsilon_{1\mu}\varepsilon_{2\nu}\varepsilon'_{1\mu'}\varepsilon'_{2\nu'}\Gamma^{\mu\nu'\rho}(p_1-p_2',p_2'-p_1)\Delta_{\rho\sigma}(p_1-p_2')\Gamma^{\mu'\nu\sigma}(-p_1',p_2,p_1'-p_2)$$
$$= \frac{g^2}{(2p^2(1+\cos\theta) - m^2)m^4}\Big(4p^2E^2(1+\cos\theta)^2(4p^2+2E^2(1-\cos\theta))$$
$$- 8p^2E^2(1+\cos\theta)(p^2+E^2\cos\theta)(3-\cos\theta)$$
$$+ (p^2+E^2\cos\theta)^2(4E^2+2p^2(1-\cos\theta))\Big)$$

$$= g^2\left(\left(\frac{E}{m}\right)^2(3-\cos\theta)(1+\cos\theta) + \left(\frac{E}{m}\right)^2\left(-\frac{3}{2} - \frac{15}{2}\cos\theta\right) + O(1)\right)$$

$$T_4 = g^2\varepsilon_{1\mu}\varepsilon_{2\nu}\varepsilon'_{1\mu'}\varepsilon'_{2\nu'}C^{\mu\nu\mu'\nu'}$$

$$= g^2\left(\left(\frac{E}{m}\right)^4(\cos^2\theta - 6\cos\theta - 3) + \left(\frac{E}{m}\right)^2(6\cos\theta + 2) + O(1)\right)$$

and so

$$T_{tot} = g^2\left(\left(\frac{E}{m}\right)^2\left(\frac{1}{2} - \frac{1}{2}\cos\theta\right) + O(1)\right)$$

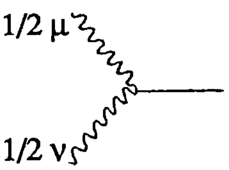
and thus the unitarity bound is violated.

By way of comparison, if masses are obtained through the Higgs mechanism, then $A_\mu{}^3$ is massless so that

$$T_s = g^2\left(4\cos\theta\left(\frac{E}{m}\right)^4 + O(1)\right)$$

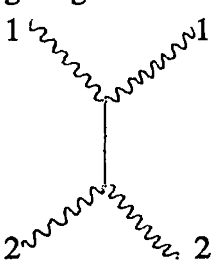
$$T_u = g^2\left(\left(\frac{E}{m}\right)^4(3-\cos\theta)(1+\cos\theta) - \left(\frac{E}{m}\right)^2 8\cos\theta + O(1)\right)$$

with T_4 as before. Further there is a coupling of the Higgs scalar to $A_\mu{}^1$ and $A_\mu{}^2$ with vertex



$$g\Gamma_{\mu\nu} = 2gm_H\eta_{\mu\nu}$$

giving a contribution to T_{tot}



$$T_H = g^2\left(\left(\frac{E}{m}\right)^2(2\cos\theta - 2) + O(1)\right)$$

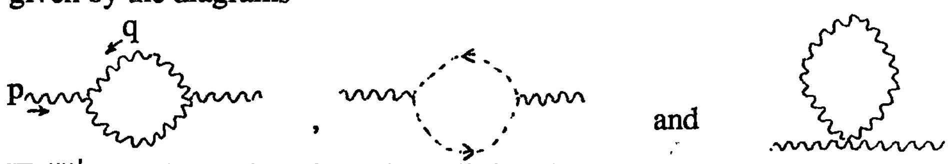
which is exactly what is required so that T_{tot} should be of order 1 and so satisfy the unitarity bound.

2. Further confirmation of the violation of unitarity is given by studying the self-energy of the vector boson [2].

$$\Pi^{\mu\mu'}_{aa'}(p) = \frac{p^\mu p^{\mu'}}{p^2}\Pi^{long}_{aa'}(p^2) + \left(\eta^{\mu\mu'} - \frac{p^\mu p^{\mu'}}{p^2}\right)\Pi^{tr}_{aa'}(p^2)$$

The imaginary part of Π^{tr} on shell, $p^2 = m^2$, $p^0>0$, gives the decay width of the physical spin-1 states. But if the theory does only describe massive spin-1 bosons, then this should be zero as there are no states into which decay is possible. More importantly the negative-norm contributions from the ghosts must be fully cancelled if the theory is to be unitary.

Now to order g^2 the contributions to $\Pi^{\mu\mu'}$ are $\Pi_V{}^{\mu\mu'}$, $\Pi_G{}^{\mu\mu'}$ and $\Pi_T{}^{\mu\mu'}$ given by the diagrams



$\Pi_V{}^{\mu\mu'}$ may be broken down by splitting the vector propagator into spin-1 and spin-0 parts

$$\Delta^{aa'}_{\mu\nu}(k) = \frac{-\eta_{\mu\nu} + \dfrac{k_\mu k_\nu}{k^2 + i\varepsilon}}{k^2 - m^2 + i\varepsilon}\, i\delta^{aa'}$$

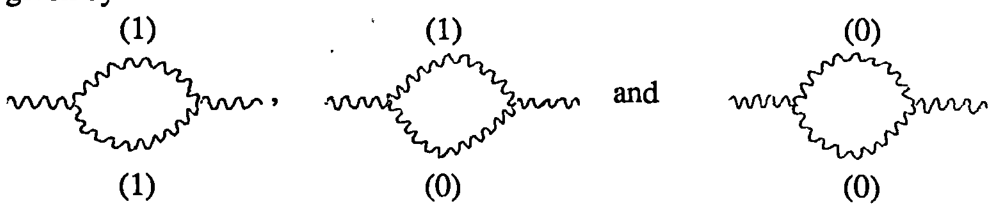
$$= \frac{-\eta_{\mu\nu} + \dfrac{k_\mu k_\nu}{m^2}}{k^2 - m^2 + i\varepsilon}\, i\delta^{aa'} - \frac{\dfrac{k_\mu k_\nu}{m^2}}{k^2 + i\varepsilon}\, i\delta^{aa'}$$

$$\equiv \Delta_{\mu\nu}^{(1)aa'}(k) + \Delta_{\mu\nu}^{(0)aa'}(k)$$

Then it becomes

$$\Pi_V^{\mu\mu'} = \Pi^{\mu\mu'}_{(1)(1)} + \Pi^{\mu\mu'}_{(0)(1)} + \Pi^{\mu\mu'}_{(0)(0)}$$

given by



The imaginary part of each contribution is given by the Cutkosky rules [8]. The tadpole diagram has no cut and so may be ignored. The imaginary part of $\Pi_{(1)(1)}$ is

$$2\mathrm{Im}\left(\Pi^{\mu\mu'}_{(1)(1)aa'}(p)\right) = \frac{g^2}{2}\int\frac{d^4q}{(2\pi)^4}\,\Gamma^{\mu\nu\rho}_{abc}(p,q,-p-q)\,\delta^{bb'}\left(-\eta_{\nu\nu'} + \frac{q_\nu q_{\nu'}}{m^2}\right)2\pi\,\theta(-q^0)\,\delta(q^2-m^2)$$
$$\delta^{cc'}\left(-\eta_{\rho\rho'} + \frac{(p+q)_\rho(p+q)_{\rho'}}{m^2}\right)2\pi\,\theta(p^0+q^0)\,\delta((p+q)^2-m^2)$$
$$(-1)\Gamma^{\mu'\nu'\rho'}_{a'b'c'}(-p,-q,p+q)$$

We need not evaluate this because here the step functions and δ-functions ensure that the whole must vanish on shell, $p^2 = m^2$, $p^0>0$. This is most easily seen in the centre of mass frame, $p^\mu = (m,0,0,0)$, for then $q^0<0$, $q^2 = m^2$, implies that

$q^0 \leq -m$ while $p^0+q^0>0$, $(p+q)^2 = m^2$ implies that $p^0+q^0 \geq m$ and so $q^0 \geq 0$, and the two are not compatible. In fact complete evaluation of $\mathrm{Im}(\Pi_{(1)(1)}(p))$ yields the factor $\theta(p^0)\theta(p^2-4m^2)$ as we would expect for this part and in accordance with unitarity.

The imaginary part of $\Pi_{(1)(0)}{}^{\mu\mu'}$ also vanishes on shell if contracted with

$$\frac{1}{3}\left(\eta_{\mu\mu'} - \frac{p_\mu p_{\mu'}}{m^2}\right) \tag{11}$$

so as to extract the transverse part. Specifically

$$\begin{aligned}
2\mathrm{Im}\left(\Pi^{\mu\mu'}_{(1)(0)aa'}(p)\right) = g^2 \int \frac{d^4q}{(2\pi)^4} \Gamma^{\mu\nu\rho}_{abc}(p,q,-p-q)\ \delta^{bb'}\left(-\frac{q_\nu q_{\nu'}}{m^2}\right) 2\pi\ \theta(-q^0)\delta(q^2) \\
\delta^{cc'}\left(-\eta_{\rho\rho'} + \frac{(p+q)_\rho (p+q)_{\rho'}}{m^2}\right) 2\pi\ \theta(p^0+q^0)\delta((p+q)^2-m^2) \\
(-1)\Gamma^{\mu'\nu'\rho'}_{a'b'c'}(-p,-q,p+q)
\end{aligned}$$

$$\begin{aligned}
= -g^2 f_{abc} f_{a'}{}^{bc} \int \frac{d^4q}{(2\pi)^2} \left(q^\mu q^\rho + p^\mu q^\rho + p^\rho q^\mu + \eta^{\rho\mu}(p^2 - (p+q)^2)\right) \\
\frac{1}{m^2}\theta(-q^0)\delta(q^2)\theta(p^0+q^0)\delta((p+q)^2 - m^2) \\
\left(-\eta_{\rho\rho'} + \frac{(p+q)_\rho (p+q)_{\rho'}}{m^2}\right) \\
\left(q^{\mu'} q^{\rho'} + p^{\mu'} q^{\rho'} + p^{\rho'} q^{\mu'} + \eta^{\rho'\mu'}(p^2 - (p+q)^2)\right)
\end{aligned}$$

which with $p^2 = m^2$ and the contraction with (11) understood, so that terms involving p_μ or $p_{\mu'}$ may be dropped, is

$$\begin{aligned}
= \frac{g^2}{m^2} f_{abc} f_{a'}{}^{bc} \int \frac{d^4q}{(2\pi)^2} \theta(-q^0)\theta(p^0+q^0)\delta(q^2)\delta((p+q)^2 - m^2) \\
q^\mu (p+q)^\rho \left(-\eta_{\rho\rho'} + \frac{(p+q)_\rho (p+q)_{\rho'}}{m^2}\right) q^{\mu'} (p+q)^{\rho'}
\end{aligned}$$

$$= 0$$

There remains $\Pi_{(0)(0)}{}^{\mu\mu'}$ and $\Pi_G{}^{\mu\mu'}$. Now, with $p^2 = m^2$ and (11) understood once again

$$2\mathrm{Im}\left(\Pi^{\mu\mu'}_{(0)(0)aa'}(p)\right) = \frac{1}{2}\, g^2 \int \frac{d^4q}{(2\pi)^4}\, \Gamma^{\mu\nu\rho}_{abc}(p,q,-p-q)\ \delta^{bb'}\left(-\frac{q_\nu q_{\nu'}}{m^2}\right) 2\pi\ \theta(-q^0)\delta(q^2)$$

$$\delta^{cc'}\left(-\frac{(p+q)_\rho (p+q)_{\rho'}}{m^2}\right) 2\pi\ \theta(p^0+q^0)\delta((p+q)^2)$$

$$(-1)\Gamma^{\mu'\nu'\rho'}_{a'b'c'}(-p,-q,p+q)$$

$$= \frac{1}{2}\frac{g^2}{m^4} f_{abc} f_{a'}{}^{bc} \int \frac{d^4q}{(2\pi)^2}\ \left(q^\mu (p+q)^2 + (p+q)^\mu (p^2 - (p+q)^2)\right)$$

$$\theta(-q^0)\delta(q^2)\theta(p^0+q^0)\delta((p+q)^2)$$

$$\left(q^{\mu'}(p+q)^2 + (p+q)^{\mu'}(p^2 - (p+q)^2)\right)$$

$$= \frac{1}{2}\, g^2 f_{abc} f_{a'}{}^{bc} \int \frac{d^4q}{(2\pi)^2}\, \theta(-q^0)\delta(q^2)\theta(p^0+q^0)\delta((p+q)^2)\ q^\mu q^{\mu'}$$

and

$$2\mathrm{Im}\left(\Pi^{\mu\mu'}_{G\,aa'}(p)\right) = -\, g^2 \int \frac{d^4q}{(2\pi)^4}\, \Gamma^{\mu}_{acb}(p+q)\ \delta^{bb'} 2\pi\ \theta(-q^0)\delta(q^2)$$

$$\delta^{cc'}\, 2\pi\ \theta(p^0+q^0)\delta((p+q)^2)\ (-1)\Gamma^{\mu'}_{a'b'c'}(q)$$

$$= -\, g^2 f_{abc} f_{a'}{}^{bc} \int \frac{d^4q}{(2\pi)^2}\, \theta(-q^0)\theta(p^0+q^0)\delta(q^2)\delta((p+q)^2)\ q^\mu q^{\mu'}$$

Thus we see that these terms are of the same sign and of the same form, but that $\Pi_{(0)(0)}{}^{\mu\mu'}$ only cancels half of $\Pi_G{}^{\mu\mu'}$. This factor of $1/2$ in (0)(0) is the symmetry factor for the internal lines; of course there is no corresponding factor for the directed ghost lines.

Continuuing we could confirm the violation of unitarity in other processes [3] and at higher orders but the fact of it is already clear. To understand when the violation of unitarity came about let us return to the unrenormalizable lagrangian (8)+(9).

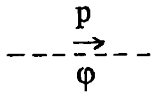
4. Renormalizability vs Unitarity

We have seen that the renormalizable lagrangian (10), which did not possess an invariance under a nilpotent BRST transformation, does not give a unitary theory. We will examine the same processes for the unrenormalizable lagrangian (8)+(9) in the Landau gauge,

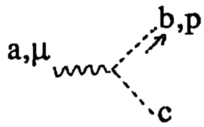
$$L = -\frac{1}{4}\,\mathrm{tr}\,F^2 + \frac{1}{2}m^2\,\mathrm{tr}\,A^2 + \frac{1}{2}\partial^\mu\varphi D_\mu\varphi + B\partial\cdot A + \overline{\omega}\partial\cdot D\omega + O(g^2)\,,$$

which does possess a nilpotent BRST symmetry.

To the Feynman rules before we must add the rules for the φ-field.



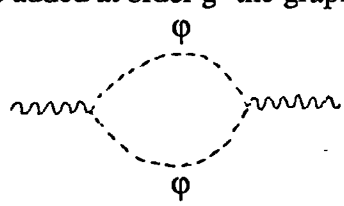
$$\frac{1}{p^2 + i\varepsilon}\, i\delta_{aa'}$$



$$gp^\mu f_{abc}$$

together with an infinite number of other vertices with more than two attached φ-lines and a corresponding higher order of g.

Then to the second process which we considered, the self energy of the vector boson, there should be added at order g^2 the graph



Now the Feynman rules for φ at this order are exactly those for the ghost field except that, as the lines are not directed, a symmetry factor must be included where appropriate and, as φ is bosonic, there is no factor of (-1) for φ loops. Thus the contribution of this graph to the imaginary part of Π_{tr} will be exactly half of that of the ghosts but with opposite sign. That is exactly what is required to restore unitarity at this order.

We can see this also by examining the respective contributions of φ and the ghosts to the effective action at this order.

$$e^{iW} = N\int D\Phi\, e^{i\int d^4x\, L(\Phi)}$$

So the ghost term in the lagrangian

$$\overline{\omega}\partial\cdot D\omega\,,$$

after integrating over the complex Grassmannian ghost fields, leads to a term

$$\ln(\det(\partial\cdot D)) = \mathrm{tr}(\ln(\partial\cdot D))$$

in the effective action, while the term

$$\varphi\partial\cdot D\varphi$$

contributes

$$\ln(\det^{-\frac{1}{2}}(\partial\cdot D)) = -\frac{1}{2}\,\mathrm{tr}(\ln(\partial\cdot D))$$

Presumably the terms of higher order in φ serve to remedy any other incomplete cancellations of the ghost contributions which might occur at higher orders.

On the other hand the first process that we examined, the elastic scattering of two spin-1 bosons, is not altered at tree level by the reinclusion of the φ-field. The amplitude continues to violate the unitarity bound. This is because the derivation of the unitarity bound also assumes the renormalizability of the theory, and this now fails to hold.

The conclusion which we must draw is that we cannot construct a consistent theory of massive, non-abelian, vector bosons without invoking the Higgs mechanism. Our attempts to ensure renormalizability were only at the expense of unitarity and vice versa. Despite this the non-abelian generalization of the Stueckelberg formulation is still interesting in the way that it demonstrates that the original naive lagrangian is of a form sufficient to ensure unitarity, and in the way in which the cancellations necessary for this unitarity are exhibited at each order.

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Conclusion

I have in this thesis considered five different topics in each of which Grassmannian variables occur in some guise.

In Chapter 2 I found a square root of the Dirac equation using the superfield formulation of space-time supersymmetry. Analysis of the massless case showed it to be equivalent to a theory involving a massless spinor obeying the Dirac equation and a complex vector field obeying Maxwell's equations. The massive case and possible interactions remain to be investigated.

In the next chapter I considered the derivation of the index for the twisted Dirac operator through path integrals in a supersymmetric quantum mechanics. By considering carefully the construction of these path integrals I was able to show that the ambiguities which arise come from taking the discrete to continuum limit. Establishing a priori certain facts about the path integrals involved in the index calculation then enabled me to eliminate these ambiguities, including the overall normalization. Also, in the first part of this chapter, I used the general Atiyah-Singer index theorem result, that the index of an operator (which is elliptic, etc.) is dependent only upon the spaces between which it acts, to derive a general expression for the indices of operators between fields of arbitrary spin in the presence of a background gravitational field. This result of the index theorem that I used here has not been clearly demonstrated in the path integral derivations of the indices of individual operators as presented so far and is perhaps worthy of consideration in the future.

Grand unified theories within the Grassmannian Kaluza-Klein scheme, which involves extra anticommuting coordinates rather than commuting ones, were the subject of Chapter 4. I showed that the SU(5) grand unified theory led to a more economical scheme than did the SO(10) one, requiring only 10 extra coordinates as opposed to 64. These investigations were conducted within the framework of the Grassmannian Kaluza-Klein ansatz. For general Grassmannian Kaluza-Klein schemes it remains to be considered whether the full (4+N)-dimensional supergravity theory can lead to the ansatz in such a way that the unwanted modes are rendered harmless, and whether the OSp(4/N) spinor representations might be tamed.

Chapter 5 was concerned with the BRST supersymmetry. It would seem that a modification of the Faddeev-Popov prescription to include the anti-BRST symmetry leads to a natural framework in which to describe the complete set of fields in a quantum theory. Further the linking of the two symmetries by an Sp(2) transformation can provide a rationale for the occurrence of ghost fields in other than ghost-antighost pairs. I gave a formulation imposing such an Sp(2)-symmetric extended-BRST supersymmetry through a (4+2)-dimensional superspace. By not attempting to treat the full (4+2)-dimensional superspace as fundamental problems in dealing with spinors, which have arisen in other such schemes, were avoided. It would now be interesting to go on to consider whether the Sp(2) symmetry might be brought naturally and profitably into the approach of Fradkin et al. to the canonical quantization of gauge theories.

Finally in Chapter 6, I discussed the renormalizability and unitarity of the massive Yang-Mills theory without Higgs. I demonstrated that a scheme based on the Stueckelberg model which guarantees renormalizability does so only at the expense of unitarity. This violation of unitarity is evident perturbatively in the failure of the theory to prevent its ghost modes from appearing in the outgoing asymptotic states. The Higgs mechanism, for which such difficulties do not occur, thus emerges strengthened, though until direct experimental evidence of the Higgs is obtained investigations into possible alternatives should surely continue.

Appendix

The purpose of this appendix is to explain the two component spinor notation used in Chapter 2 and to give some useful identities.. The notation is that of Wess and Bagger, except that they use m,n,etc. for vector indices while we use μ,ν,etc., and their metric differs from ours by an overall sign; that is we take as the Minkowski metric (as throughout this thesis)

$$\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$$

with μ,ν,etc. running over 0,1,2,3. The identities given here can also be found in the appendices of their book.

Undotted Greek letters at the beginning of the alphabet, α,β,etc., are spinor indices corresponding to the $(^1/_2,0)$ representation of the Lorentz group, dotted ones, $\dot\alpha$,$\dot\beta$,etc., are spinor indices corresponding to the conjugate representation $(0,^1/_2)$, they all take the values 1 or 2.

In accordance with spin-statistics, objects with an odd number of spinor indices usually anticommute and objects with an even number commute.

The antisymmetric tensors $\varepsilon^{\alpha\beta}$, $\varepsilon_{\alpha\beta}$, $\varepsilon^{\dot\alpha\dot\beta}$, $\varepsilon_{\dot\alpha\dot\beta}$, with $\varepsilon^{12}=1$, $\varepsilon_{12}=-1$, are used to raise and lower indices from the left, i.e.

$$\psi^{\alpha} = \varepsilon^{\alpha\beta}\psi_{\beta}\,,\ \ \overline{\psi}_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\overline{\psi}^{\dot\beta}\,,\ \text{etc.}$$

Contracting indices gives a scalar. We take

$$\psi\chi = \psi^{\alpha}\chi_{\alpha} = \chi^{\alpha}\psi_{\alpha} = \chi\psi$$

$$\overline{\psi\chi} = \overline{\psi}_{\dot\alpha}\overline{\chi}^{\dot\alpha} = \overline{\chi}_{\dot\alpha}\overline{\psi}^{\dot\alpha} = \overline{\chi\psi}$$

In the chiral representation a Dirac spinor is

$$\Psi = \begin{pmatrix} \chi_{\alpha} \\ \overline{\psi}^{\dot\alpha} \end{pmatrix}$$

and the Dirac matrices are

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \overline{\sigma}^{\mu} & 0 \end{pmatrix},\ \ \Gamma = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $\sigma^1,\sigma^2,\sigma^3$ are the Pauli matrices and $\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and

$$\overline{\sigma}^{\mu\dot\alpha\alpha} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}\sigma^{\mu}_{\beta\dot\beta} = -(\varepsilon\sigma^{\mu}\varepsilon)^{\alpha\dot\alpha}$$

So

$$(\varepsilon\overline{\sigma}^{\mu})_{\dot\alpha}{}^{\alpha} = (\varepsilon\sigma^{\mu})^{\alpha}{}_{\dot\alpha}$$

$$(\overline{\sigma}^{\mu}\varepsilon)^{\dot\alpha}{}_{\alpha} = (\sigma^{\mu}\varepsilon)_{\alpha}{}^{\dot\alpha}$$

$$\mathrm{tr}(\sigma^{\mu}\overline{\sigma}^{\nu}) = 2\eta^{\mu\nu}$$

$$\sigma^{\mu}_{\alpha\dot\alpha}\overline{\sigma}_{\mu}^{\dot\beta\beta} = 2\delta_{\alpha}^{\beta}\delta_{\dot\alpha}^{\dot\beta}$$

$$(\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu)_\alpha{}^\beta = 2\eta^{\mu\nu}\delta_\alpha^\beta$$

$$(\bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu)^{\dot\alpha}{}_{\dot\beta} = 2\eta^{\mu\nu}\delta^{\dot\alpha}_{\dot\beta}$$

$$\sigma^\mu\bar\sigma^\nu\sigma^\rho + \sigma^\rho\bar\sigma^\nu\sigma^\mu = 2(\eta^{\mu\nu}\sigma^\rho + \eta^{\nu\rho}\sigma^\mu - \eta^{\mu\rho}\sigma^\nu)$$

$$\bar\sigma^\mu\sigma^\nu\bar\sigma^\rho + \bar\sigma^\rho\sigma^\nu\bar\sigma^\mu = 2(\eta^{\mu\nu}\bar\sigma^\rho + \eta^{\nu\rho}\bar\sigma^\mu - \eta^{\mu\rho}\bar\sigma^\nu)$$

$$\bar\sigma^\mu\sigma^\nu\bar\sigma^\rho - \bar\sigma^\rho\sigma^\nu\bar\sigma^\mu = 2i\varepsilon^{\mu\nu\rho\sigma}\bar\sigma_\sigma$$

$$\sigma^\mu\bar\sigma^\nu\sigma^\rho - \sigma^\rho\bar\sigma^\nu\sigma^\mu = -2i\varepsilon^{\mu\nu\rho\sigma}\sigma_\sigma$$

and, defining

$$\sigma^{\mu\nu} = \tfrac{1}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)$$

$$\bar\sigma^{\mu\nu} = \tfrac{1}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu),$$

$$(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta} = (\sigma^{\mu\nu}\varepsilon)_{\beta\alpha}$$

$$(\bar\sigma^{\mu\nu}\varepsilon)^{\dot\alpha\dot\beta} = (\bar\sigma^{\mu\nu}\varepsilon)^{\dot\beta\dot\alpha}, \text{ etc.}$$

$$\tfrac{i}{2}\varepsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} = \sigma^{\mu\nu}$$

$$\tfrac{i}{2}\varepsilon^{\mu\nu\rho\sigma}\bar\sigma_{\rho\sigma} = -\bar\sigma^{\mu\nu}$$

$$\mathrm{tr}(\sigma^{\mu\nu}\sigma^{\rho\sigma}) = -\tfrac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) - \tfrac{i}{2}\varepsilon^{\mu\nu\rho\sigma}$$

An antisymmetric two-index object must be proportional to ε, so

$$\psi^\alpha\psi^\beta = -\tfrac{1}{2}\varepsilon^{\alpha\beta}\psi\psi$$

$$\bar\psi^{\dot\alpha}\bar\psi^{\dot\beta} = \tfrac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\overline{\psi\psi}, \text{ etc.}$$

Since complex conjugation reverses the order of factors, its effect on derivatives is

$$\overline{\frac{\partial}{\partial\theta^\alpha}F(x,\theta,\bar\theta)} = -(-1)^F\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\overline{F(x,\theta,\bar\theta)}$$

where $F = 0$ or 1 according as $F(x,\theta,\theta)$ is overall commuting or anticommuting. An example illustrates this

$$\frac{\partial}{\partial\theta^\alpha}\theta^\beta\chi_\beta = \chi_\alpha$$

$$\overline{\frac{\partial}{\partial\theta^\alpha}\theta^\beta\chi_\beta} = \bar\chi_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}}\bar\theta^{\dot\beta}\bar\chi_{\dot\beta} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\overline{\theta\chi}$$

whereas

$$\frac{\partial}{\partial\theta^\alpha}\theta^\beta m = m\delta_\alpha^\beta$$

$$\overline{\frac{\partial}{\partial\theta^\alpha}\theta^\beta m} = \bar m\delta_{\dot\alpha}^{\dot\beta} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}}\bar\theta^{\dot\beta}\bar m = \frac{\partial}{\partial\bar\theta^{\dot\alpha}}\overline{\theta^\beta m}$$

Thus, since

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \quad \text{and} \quad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu ,$$

$$\overline{D_\alpha F} = (-1)^F \bar{D}_{\dot{\alpha}}\bar{F} .$$